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Quantum Gravity Seminar
HW#2 — Action as a Functor

1) Let A be a smooth 1-form on X. We define a functor

$$S: P(X) \longrightarrow \mathbb{R}$$

by letting

$$S(\gamma) = \int_\gamma A$$

for any morphism γ in $P(X)$. Uniqueness of such a functor is immediate, since we have a well defined formula for the morphism map and since the group $\mathbb{R}$ has only one object. To see that this actually gives a functor, suppose

$$\gamma_1 : [0, T_1] \longrightarrow X \quad \& \quad \gamma_2 : [0, T_2] \longrightarrow X$$

are piecewise smooth paths in X with $\gamma_1(T_1) = \gamma_2(0)$ so they are composable as morphisms in $P(X)$. Then $\gamma_1\gamma_2 : [0, T_1 + T_2]$ is given by

$$\gamma_1\gamma_2(t) = \begin{cases} \gamma_1(t) & t \in [0, T_1] \\ \gamma_2(t - T_1) & t \in [T_1, T_1 + T_2] \end{cases},$$

so:

$$\begin{aligned} S(\gamma_1\gamma_2) = \int_{\gamma_1\gamma_2} A &= \int_0^{T_1+T_2} A((\gamma_1\gamma_2)'(t))\, dt \\ &= \int_0^{T_1} A((\gamma_1\gamma_2)'(t))\, dt + \int_{T_1}^{T_1+T_2} A((\gamma_1\gamma_2)'(t))\, dt \\ &= \int_0^{T_1} A(\gamma_1'(t))\, dt + \int_0^{T_2} A(\gamma_2'(t))\, dt \\ &= S(\gamma_1) + S(\gamma_2), \end{aligned}$$

and also

$$S([0,0] \xrightarrow{1_x} X) = \int_0^0 A(1_x') = 0.$$

2) The bundle projection

$$\pi : T^*\mathcal{U} \longrightarrow \mathcal{U}$$
$$(q,p) \longmapsto q$$

yields

$$(d\pi)_{(q,p)} : T_{(q,p)}(T^*\mathcal{U}) \longrightarrow T_q\mathcal{U} .$$

Using the orthonormal basis $\left\{\left(\frac{\partial}{\partial x_i}\right)_q\right\}$ of $T_q\mathcal{U}$, we may expand $d\pi_{(q,p)}(v)$, $v \in T_{(q,p)}(T^*\mathcal{U})$, as:

$$\begin{aligned} d\pi_{(q,p)}(v) &= (dx_i)_q\left(d\pi_{(q,p)}(v)\right)\left(\frac{\partial}{\partial x_i}\right)_q \\ &= (dx_i)_q \circ (d\pi)_{(q,p)}(v)\left(\frac{\partial}{\partial x_i}\right)_q \quad \text{chain Rule} \\ &= d(x_i \circ \pi)_{(q,p)}(v)\left(\frac{\partial}{\partial x_i}\right)_q \\ &= (dq_i)_{(q,p)}(v)\left(\frac{\partial}{\partial x_i}\right)_q \end{aligned}$$

where the chain rule step made use of the fact that

$$q_i(q,p) = x_i(q) = x_i \circ \pi(q,p).$$

Thus:

$$\begin{aligned} \alpha_{(q,p)}(v) &:= p\left(d\pi_{(q,p)}(v)\right) \\ &= p\left((dq_i)_{(q,p)}(v)\left(\frac{\partial}{\partial x_i}\right)_q\right) \\ &= (dq_i)_{(q,p)}(v)\, p\left(\frac{\partial}{\partial x_i}\right)_q = (dq_i)_{(q,p)}(v)\, p^i \end{aligned}$$

Assembling this into α, we just drop all the basepoints:

$$\alpha(v) = p^i dq_i(v)$$

or since v was arbitrary:

$$\alpha = p^i dq_i \, .$$

3.) $\varpi = d\alpha = d(p^i dq_i) = dp^i \wedge dq_i + \underbrace{p^i d(dq_i)}_{=0} = dp^i \wedge dq_i \, .$

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4.) By Stokes' Thm:

$$S(\gamma) = \int_\gamma \alpha = \int_{\partial D} \alpha = \int_D d\alpha = \int_D \varpi \, .$$

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