Light! Functors! Action!

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1. We understand the real numbers to be $(\mathbb{R}, +)$, the additive group of reals, interpreted as a category in the usual way for groups (each real number gives a morphism, and composition is addition). In this case, to be a functor, the map $S : \gamma \mapsto \int_\gamma A$ should be additive: if $\gamma_1 : [0, T_1] \to X$ and $\gamma_2 : [0, T_2] \to X$ are Moore paths, then:

$$\begin{aligned} S(\gamma_1\gamma_2) &= \int_{\gamma_1\gamma_2} A \\ &= \int_{\gamma_1} A \quad + \quad \int_{\gamma_2} A \\ &= S(\gamma_1) + S(\gamma_2) \end{aligned}$$

   This is so since the integral along $\gamma_1\gamma_2$, we have two pieces, one of which is the integral along γ_1 and the other the integral along γ_2, since integration is independent of parametrization, and linear. Moreover, the integral along the zero path is zero, so $S(\mathrm{Id}_x) = 0 = \mathrm{Id}_{\mathbb{R}}$. These two properties mean that S is a functor.

2. We have a 1-form $\alpha_x : T_x(T^*M) \to \mathbb{R}$ taking a vector v, in the tangent space to the cotangent bundle at some point $x = (q, p)$, to $p(\mathrm{d}\pi(v))$ (i.e. applying the momentum covector p to the image of the vector v pushed forward along the projection map onto M). We want to show that, in terms of the coordinates x_i and the induced coordinates on the cotangent bundle, this amounts to $\alpha = \sum_i p^i \mathrm{d}q_i$. Consider the effect of this 1-form on the (generic) vector v:

$$\begin{aligned} \left(\sum_i p^i \mathrm{d}q_i\right)(v) &= \sum_i p\left(\frac{\partial}{\partial x_i}\right)\mathrm{d}q_i(v) \\ &= \sum_i p\left(\frac{\partial}{\partial x_i}\right) v(x_i(q)) \qquad \text{(by definition of } p^i, q_i, \text{and d)} \\ &= v\left(\sum_i p\left(\frac{\partial}{\partial x_i}\right) x_i(q)\right) \quad \text{(by linearity of } v\text{)} \end{aligned}$$

   so that we have

$$\sum_i p^i \mathrm{d}q_i \quad = \quad \sum_i p\left(\frac{\partial}{\partial x_i}\right) x_i(q)$$

   But this (real) number is just the representation in the basis given by the $\frac{\partial}{\partial x_i}$ vectors of the effect of p on the part of v in the subspace of $T_x(T^*M)$ corresponding to the position coordinates on M.

Now, $p(\mathrm{d}\pi(v))$ is a real number which arises by taking a tangent vector v to T^*M and projecting by the differential of π to a tangent vector to M (i.e. if the tangent vector v corresponds to differentiation along a curve, then $\mathrm{d}\pi(v)$ corresponds to differentiation along the projection onto M) and then applying p. This is just the same as the description just given for $\sum_i p^i \mathrm{d}q_i$. So in fact, as we wanted, $\alpha = \sum_i p^i \mathrm{d}q_i$.

3. We have that $\alpha = p^i \mathrm{d}q_i$ (now using the Einstein convention), so if $\omega = \mathrm{d}\alpha$, we have

$$\begin{aligned} \omega &= \mathrm{d}\alpha \\ &= \mathrm{d}(p^i \mathrm{d}q_i) \\ &= \mathrm{d}p^i \wedge \mathrm{d}q_i + p^i \wedge \mathrm{d}(\mathrm{d}q_i) \quad && \text{(Leibniz rule for d)} \\ &= \mathrm{d}p^i \wedge \mathrm{d}q_i && (\mathrm{d}^2 = 0) \end{aligned}$$

(note that in each case, here, we actually have a sum over i, where these terms appear for each i).

4. Supposing we have a disc D with boundary γ. Then the action for γ is $S(\gamma) = \int_\gamma \alpha$. But by Stokes' Theorem,

$$S(\gamma) = \int_\gamma \alpha = \int_{\partial D} \alpha \quad = \int_D \mathrm{d}\alpha = \int_D \omega$$