Action as a functor

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1. *Connections as functors.*

The functor S must map $\gamma\colon p \to q$ to $S(\gamma)\colon S(p) \to S(q)$, where $S(p)$ and $S(q)$ are objects in $\mathbf{R}$. But $\mathbf{R}$ is a category with one object only, so $S(p) = S(q) = \bullet$ and the functor S is uniquely determined by

$$S(\gamma) = \int_\gamma A.$$

To see that this is a functor, one need only recall from differential geometry that

$$\int_{\gamma\circ\gamma'} A = \int_\gamma A + \int_{\gamma'} A.$$

[Thanks to Derek for pointing out that the point of this exercise is the triviality of the object map.]

2. *The symplectic potential in local coordinates.*

I prefer the notation $p_q \in T^*_p M \subseteq T^*M$ rather than (q,p). The coordinate functions q_i and p^i on T^*M are such that

$$q_i(p_q) = x_i(q), \qquad \text{and} \quad p^i(p_q) = p_q\Big(\frac{\partial}{\partial x_i}\Big|_q\Big)$$

A tangent vector $v_x \in T(T*M)$, where $x = p_q \in T^*M$ (beware of the unfortunate clash of notations between $x \in T^*M$ and $x_i\colon M \to \mathbf{R}$), is of the form

$$v_x = v_i \, \frac{\partial}{\partial q_i}\Big|_x + v^j \, \frac{\partial}{\partial p^j}\Big|_x .$$

The map $d\pi\colon T(T^*M) \to TM$ maps $T_x(T^*M) \to T_{\pi(x)}M$, and its differential, and in fact

$$d\pi(v) = v^i \, \frac{\partial}{\partial x_i}\Big|_{\pi(x)} .$$

By definition of p^i and of v, then,

$$\alpha(v) = \sum_i p^i v_i = \sum_i p^i dq^i(v).$$

3. *The symplectic structure.*

The exterior derivative of

$$\alpha = p^i dq_i$$

is

$$d\alpha = dp^i \wedge dq_i.$$

4. *The action as phase-space area.*

By the generalized Stokes' theorem, if $\gamma = \partial D$,

$$S(\gamma) = \oint_\gamma \alpha = \oint_{\partial D} \alpha = \int_D d\alpha = \int_D \omega.$$