Action as a Functor

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In class I said some mysterious things about how the electromagnetic field is a U(1) *connection on spacetime, while the Feynman path integral involves a* U(1) *connection on the cotangent bundle of spacetime. Let's clarify this a little.*

Given a smooth manifold X, let $\mathcal{P}(X)$ be the **category of paths** *in X. There are lots of different ways to make this precise. I sketched one in class; here's another, where the interval parametrizing the paths can be of arbitrary length. Such paths are called 'Moore paths', but they may actually have been introduced by the topologist R. L. Moore.*

The objects of $\mathcal{P}(X)$ are points of X. A morphism in $\mathcal{P}(X)$, say $\gamma\colon x \to y$, is a piecewise smooth path $\gamma\colon [0,T] \to X$ with

$$\gamma(0) = x, \qquad \gamma(T) = y,$$

where $T \geq 0$ is arbitrary. The composite of a path $\gamma_1\colon [0,T_1] \to X$ and a path $\gamma_2\colon [0,T_2] \to X$ is a path $\gamma_1\gamma_2\colon [0,T_1+T_2] \to X$, defined in the obvious way:

$$(\gamma_1\gamma_2)(t) = \begin{cases} \gamma_1(t) & 0 \leq t \leq T_1 \\ \gamma_2(t-T_1) & T_1 \leq t \leq T_1+T_2 \end{cases}$$

It's easy to check the associative law and left/right unit laws with this definition, where the identity path 1_x is the path $\gamma\colon [0,0] \to X$ with $\gamma(0) = x$.

Now let us see how a 1-form on X defines a notion of $\mathbb{R}$-valued or U(1)*-valued parallel transport. To do this, we think of the groups $\mathbb{R}$ and* U(1) *as 1-object categories with all morphisms invertible.*

1. Suppose that A is a smooth 1-form on X. Show that there's a unique functor

$$S\colon \mathcal{P}(X) \to \mathbb{R}$$

with

$$S(\gamma) = \int_\gamma A$$

for any morphism γ of $\mathcal{P}(X)$.

Note that the definition of $\int$ is

$$\int_\gamma A = \int_0^T \langle A(\gamma(t))|\gamma'(t)\rangle \,\mathrm{d}t.$$

(I'm using kets for tangent vectors and bras for cotangent vectors.) Now, to be a functor, I must have $S(1_x) = 0$ and $S(\gamma_1\gamma_2) = S(\gamma_1) + S(\gamma_2)$. Indeed,

$$S(1_x) = \int_{1_x} A = \int_0^0 \textit{whatever}\,\mathrm{d}t = 0.$$

Also,

$$S(\gamma_1\gamma_2) \;=\; \int_{\gamma_1\gamma_2} A = \int_0^{T_1+T_2} \langle A((\gamma_1\gamma_2)(t))|(\gamma_1\gamma_2)'(t)\rangle \,\mathrm{d}t$$

$$
\begin{aligned}
&= \int_0^{T_1} \langle A(\gamma_1(t))|\gamma_1'(t)\rangle \,\mathrm{d}t + \int_{T_1}^{T_1+T_2} \langle A(\gamma_2(t-T_1))|\gamma_2'(t-T_1)\rangle \frac{\mathrm{d}(t-T_1)}{\mathrm{d}t}\,\mathrm{d}t \\
&= \int_0^{T_1} \langle A(\gamma_1(t))|\gamma_1'(t)\rangle \,\mathrm{d}t + \int_0^{T_2} \langle A(\gamma_2(t))|\gamma_2'(t)\rangle \,\mathrm{d}t \\
&= \int_{\gamma_1} A + \int_{\gamma_2} A = S(\gamma_1) + S(\gamma_2).
\end{aligned}
$$

Therefore, this is indeed a functor. (Note that in this calculation, it is irrelevant that $\frac{\mathrm{d}(t-T_1)}{\mathrm{d}t} = 1$*. This expression appears from differentiating* $\gamma_1\gamma_2$*, and it disappears in the integral's change of variable. Any smooth parametrisation would do the same.)*

In the last homework you saw that group homomorphisms are actually functors. Thus, we can compose the above functor S *with the homomorphism* $t \mapsto \exp(it)$ *from* $\mathbb{R}$ *to* $\mathrm{U}(1)$ *to get a functor from* $\mathcal{P}(X)$ *to* $\mathrm{U}(1)$*. Let's call this functor*

$$e^{iS}: \mathcal{P}(M) \to \mathrm{U}(1).$$

Next, let's show that whenever M *is a smooth manifold, the cotangent bundle* T^*M *has a god-given smooth 1-form on it. This is called the* **tautologous 1-form** *or* **symplectic potential***, and denoted by* α*.*

Recall that T^*M *is the manifold whose points are pairs* (q,p) *where* $q \in M$ *and* $p \in T_q^*M$ *is a* **cotangent vector** *at* q*, that is, a linear functional* $p: T_qM \to \mathbb{R}$ *where* T_qM *is the tangent space of* M *at* q*. The manifold* T^*M *becomes a vector bundle over* M *with projection*

$$
\begin{aligned}
\pi: T^*M &\to M \\
(q,p) &\mapsto q
\end{aligned}
$$

It is then called the **cotangent bundle** *of* M*.*

To define a 1-form α *on* T^*M*, all we need is a a linear functional* α_x *on the tangent space of each point* $x \in T^*M$*. If* $x = (q,p)$ *as above, this is given by*

$$
\begin{aligned}
\alpha_x: T_x(T^*M) &\to \mathbb{R} \\
v &\mapsto p(d\pi(v))
\end{aligned}
$$

Sneaky, huh? You should ponder this carefully until you get it.

To understand α *better, let's work out a formula for it in terms of local coordinates on* T^*M *coming from local coordinates on* M*.*

Suppose $U \subseteq M$ *is an open set in* M *equipped with coordinate functions* $x_i: U \to \mathbb{R}$*. Then the open set* $T^*U \subseteq T^*M$ *gets coordinates* $q_i, p^i: T^*U \to \mathbb{R}$ *where for any point* $(q,p) \in T^*M$*,*

$$
\begin{aligned}
q_i(q,p) &= x_i(q) \\
p^i(q,p) &= p\left(\frac{\partial}{\partial x_i}\right).
\end{aligned}
$$

Here $\frac{\partial}{\partial x_i}$ *is the tangent vector at* $q \in M$ *pointing in the* x_i *direction.*

2. In terms of the above coordinates, show that on T^*U we have

$$\alpha = \sum_i p^i dq_i$$

First notice that

$$\sum_i p^i(q,p)\left\langle \mathrm{d}x_i(q)\middle|\frac{\partial}{\partial x_j}(q)\right\rangle = \sum_i p^i(q,p)\delta_i^j = p^j(q,p) = \left\langle p\middle|\frac{\partial}{\partial x_j}(q)\right\rangle.$$

Since the vectors $|\frac{\partial}{\partial x_j}(q)\rangle$ *form a basis for* $T_q(M)$*, this proves that* $\langle p| = \sum_i p^i(q,p)\,\langle \mathrm{d}x_i(q)|$*. Next, notice that* $q_i = x_i \circ \pi$*. Thus,* $\mathrm{d}q_i = \mathrm{d}x_i \circ \mathrm{d}\pi$*, or* $\langle \mathrm{d}q_i(q,p)| = \langle \mathrm{d}x_i(q)|\mathrm{d}\pi(q,p)|$*. Putting these facts together,*

$$\langle \alpha(q,p)|v\rangle = \langle p|\mathrm{d}\pi(q,p)|v\rangle = \sum_i p^i(q,p)\,\langle \mathrm{d}x_i(q)|\mathrm{d}\pi(q,p)|v\rangle = \sum_i p^i(q,p)\,\langle \mathrm{d}q_i(q,p)|v\rangle\,.$$

Therefore, $\langle \alpha(q,p)| = \sum_i p^i(q,p)\,\langle \mathrm{d}q_i(q,p)|$*, or simply* $\alpha = \sum_i p^i\,\mathrm{d}q_i$*.*

If we use the **Einstein summation convention***, which says we always sum over indices that appear twice, once as a superscript and once as a subscript, we can abbreviate the above formula as:*

$$\alpha = p^i dq_i.$$

In case you were wondering, this is why we write the index on p^i *as a superscript.*

As a consequence of 1 *and* 2*, we see there is a god-given functor*

$$S\colon \mathcal{P}(T^*M) \to \mathbb{R}$$

given by

$$S(\gamma) = \int_\gamma \alpha$$

for any piecewise smooth path in T^*M*. We call* $S(\gamma)$ *the* **action** *of the path* γ*. We also get a functor*

$$e^{iS}\colon \mathcal{P}(T^*M) \to \mathrm{U}(1).$$

In physics we can apply these ideas by letting M *be the* **configuration space** *of a classical system, that is, the space of possible positions of the system. Then* T^*M *becomes the* **phase space***, that is, the space of possible* **states** *of the system. For a state* $(q,p) \in T^*M$*, we call* $q \in M$ *the* **position** *and* $p \in T^*_qM$ *the* **momentum***.*

The functor S *then assigns to any path in phase space a real number called its* **action***. The functor* e^{iS} *assigns to any path a phase; this becomes important in the path-integral approach to quantum mechanics.*

The phase space T^*M *also has a god-given 2-form on it, given by*

$$\omega = d\alpha$$

This is called the **symplectic structure***.*

3. Using the Einstein summation convention, show that on T^*U we have

$$\omega = dp^i \wedge dq_i$$

in terms of the previously described coordinates p^i, q_i.

Since d *is linear, it's safe to use the Einstein summation convention throughout the calculation:*

$$\omega = \mathrm{d}\alpha = \mathrm{d}(p^i\,\mathrm{d}q_i) = \mathrm{d}p^i \wedge \mathrm{d}q_i + p^i\,\mathrm{d}\mathrm{d}q_i = \mathrm{d}p^i \wedge \mathrm{d}q_i + p^i 0 = \mathrm{d}p^i \wedge \mathrm{d}q_i.$$

A path is called a **loop** *if it starts where it ends. The action of a loop in phase space is the integral of the symplectic structure over any disk having that loop as its boundary:*

4. Suppose that γ is a loop in T^*M. Suppose D is a disk in T^*M whose boundary is γ. Show that

$$S(\gamma) = \int_D \omega.$$

Since γ is the boundary of D, apply the Stokes theorem:

$$S(\gamma) = \int_\gamma \alpha = \int_{\partial D} \alpha = \int_D \mathrm{d}\alpha = \int_D \omega.$$

In an early stage of quantum mechanics Bohr and Sommerfeld thought that the 'energy eigenstates' of a quantum system corresponded to the periodic motions of the corresponding classical system in which it traced out a loop γ in phase space for which $S(\gamma)$ was a multiple of 2π, and thus $e^{iS(\gamma)} = 1$. This is a bit oversimplified, but it's still a useful idea.

Another application of these ideas to physics arises when we let M be spacetime. If a particle traces out a path in spacetime, say $\gamma\colon [0,T] \to M$, the laws governing this particle let us 'lift' this path to T^*M. In other words, they give us a formula for a path $\tilde{\gamma}\colon [0,T] \to T^*M$ such that $\pi(\tilde{\gamma}(t)) = \gamma(t)$. To be kind, I will omit this formula. The point, however, is that we can then define the action of the path γ to be

$$S(\gamma) = \int_{\tilde{\gamma}} \alpha.$$

If the particle has electric charge c and there is an electromagnetic field on spacetime, the formula for its action gets a bit fancier. An electromagnetic field is described by a 1-form A on the spacetime M. If the particle traces out a path $\gamma\colon [0,T] \to M$ in spacetime, its action is then defined to be

$$S(\gamma) = \int_{\tilde{\gamma}} \alpha + c \int_\gamma A.$$

Here we see the electromagnetic field on spacetime and the symplectic potential on the cotangent bundle of spacetime showing up in the same formula!