Groups as Categories

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Here's a little homework just to make sure you understand the concepts of *category*, *functor* and *natural transformation*, as defined in the handout 'Some Definitions Everyone Should Know'.

Recall that a set with an associative binary product and an element serving as the unit for this product is called a **monoid**: examples include $(\mathbb{N}, +, 0)$ and $(\mathbb{N}, \cdot, 1)$. A monoid where every element has a two-sided inverse is called a **group**.

A category C with only one object (say $*$) is the same thing as a monoid, since all C has is a set of morphisms $f\colon * \to *$ that can be composed associatively, together with a morphism $1_*\colon * \to *$ serving as the unit for composition.

Similarly, a category with only one object and all morphisms invertible is the same as a group!

So, among other things, category theory is a massive generalization of group theory. This means that whenever you encounter a definition in category theory, you should figure out what it amounts to in the case of groups.

In what follows, you can either do problems 1–5 or problem 6. I greatly prefer answers in LaTeX.

1. Suppose that G and H are groups, and regard them as one-object categories with all morphisms invertible. Figure out what a functor $F\colon G \to H$ amounts to. What are such functors usually called?

2. Suppose G and H are groups regarded as categories, and let $F, F'\colon G \to H$ be a pair of functors. Figure out what a natural transformation $\alpha\colon F \Rightarrow F'$ amounts to.

3. Suppose G is a group regarded as a category and let $1_G\colon G \to G$ be the identity functor. Figure out what a natural transformation $\alpha\colon 1_G \Rightarrow 1_G$ amounts to. What is the set of all such natural transformations usually called?

4. Let Vect be the category of vector spaces over your favorite field, where the morphisms are linear transformations. Suppose G is a group regarded as a category. Figure out what a functor $F\colon G \to \mathrm{Vect}$ amounts to. What is such a functor usually called?

5. Suppose G is a group regarded as a category and let $F, F'\colon G \to \mathrm{Vect}$ be functors. Figure out what a natural transformation $\alpha\colon F \Rightarrow F$ amounts to. What is such a natural transformation usually called?

6. Suppose G is a Lie group, regarded as a one-object category where the morphisms form a manifold. Let $\mathrm{Aut}(G)$ be the category whose objects are smooth invertible functors $F\colon G \to G$ and whose morphisms are smooth invertible natural transformations $\alpha\colon F \to F'$. The objects of $\mathrm{Aut}(G)$ form a Lie group. Any object F in $\mathrm{Aut}(G)$ gives a subset $[F]$ consisting all objects that are isomorphic to it. What do these subsets look like for $G = \mathrm{SO}(3)$? How about for $G = \mathrm{SU}(2)$?