Groups as Categories

Jeff Morton

October 7, 2004

1. If G and H are groups regarded as categories, then a functor $F : G \to H$ gives a map of objects (which is trivial, since there is only one object in each category, hence only one such map to choose), and a map between hom-sets of pairs of objects which correspond under the object map. Since there is only one object, this just means the functor is a map between the groups G and H considered as the hom-set $\hom(\star_G, \star_G)$ and $\hom(\star_H, \star_H)$ respectively (where the $\star$-objects are the lone objects in the categories G and H respectively). This F must satisfy
$$F(1_G) = 1_H$$
and
$$F(fg) = F(f)F(g)$$
These apply to all morphisms (group elements of G and H), and so such a functor is precisely a *homomorphism of groups*.

2. If G and H are as in the previous question, F and F' are functors (hence homomorphisms of the "underlying" groups), any natural transformation $\alpha : F \Rightarrow F'$ consists of, for each object in the category G (of which there is only one: $\star_G$), a morphism $\alpha_\star$ between $F(\star_G)$ and $F'(\star_G)$. Since there is just one object in each category and the object maps for F and F' are both the unique one, this means α amounts to a choice of a morphism in H as a category, or in other words, an element of the group H. This should make the naturality square commute, so that given any $g \in G$ (i.e. any morphism in $\hom(\star_G, \star_G)$), we should have $F(g)\alpha = \alpha F'(g)$.[1]

   This amounts to the requirement that $F(g) = \alpha F'(g)\alpha^{-1}$ - in other words, that the images of G in H by F and F' are conjugate by the group element $\alpha \in H$, and indeed we can say $F = \alpha F' \alpha^{-1}$.

3. If G is as above and $1_G : G \to G$ is the identity functor, then a natural transformation $\alpha : 1_G \Rightarrow 1_G$, by the previous result, amounts to a choice of a group element $\alpha \in G$ (where G is thought of playing the role of H in the case with two different groups), which should have the property that for each morphism g of G as a category (i.e. each element of G seen as a group), we have that it is conjugate to itself by α: that is, $g\alpha = \alpha g$. In other words,

[1] Yes: $F(g)\alpha = \alpha F'(g)$. We are using the "non-evil" convention on composition order.

such a natural transformation amount to a choice of an element of G which commutes with every $g \in G$. The set of all such things is the *centre* [2] of G.

4. If Vect is the category of vector spaces and G is a group seen as a category as above, a functor from G into Vect, $F : G \to \text{Vect}$, consists of two maps. The first is a map from the objects of G to the objects of Vect: since G has only one object, this amounts to a *choice of vector space*, say V. The second is a map from the morphisms of G (i.e. the group elements) into the linear automorphisms of V (since it is a map from $\hom(\star_G, \star_G)$ to $\hom(V, V)$). This map has to satisfy the properties that $F(1_G) = 1_V$, and $F(fg) = F(f)F(g)$. The combination of the choice of V and a map with these properties is a *representation* of G on V.

5. If G is a group as above, and $F, F' : G \to \text{Vect}$ are such functors (representations), a natural transformation $\alpha : F \Rightarrow F'$ consists of a choice, (for all objects of G - i.e., for the lone element $\star_G$), of a morphism α of Vect in $\hom(V, V')$ (the images of $\star_G$ under F and F' respectively - that is, the vector spaces of these representations). This morphism should satisfy, for every morphism $g \in G$ (i.e. for every group element), the naturality relation $F(g)\alpha = \alpha F'(g)$. This is a *homomorphism of representations* (or a *G-linear map* between V and V').

[2] Yes: the *centre*.