Groups as Categories

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Here's a little homework just to make sure you understand the concepts of *category*, *functor* and *natural transformation*, as defined in the handout 'Some Definitions Everyone Should Know'.

Recall that a set with an associative binary product and an element serving as the unit for this product is called a **monoid**: examples include $(\mathbb{N}, +, 0)$ and $(\mathbb{N}, \cdot, 1)$. A monoid where every element has a two-sided inverse is called a **group**.

A category C with only one object (say $*$) is the same thing as a monoid, since all C has is a set of morphisms $f\colon * \to *$ that can be composed associatively, together with a morphism $1_*\colon * \to *$ serving as the unit for composition.

Similarly, a category with only one object and all morphisms invertible is the same as a group!

So, among other things, category theory is a massive generalization of group theory. This means that whenever you encounter a definition in category theory, you should figure out what it amounts to in the case of groups.

In what follows, you can either do problems 1–5 or problem 6. I greatly prefer answers in LaTeX.

1. Suppose that G and H are groups, and regard them as one-object categories with all morphisms invertible. Figure out what a functor $F\colon G \to H$ amounts to. What are such functors usually called?

F consists of a function $F\colon G \to H$, such that $F(1_G) = 1_H$ and for any pair of elements f, g of G, $F(fg) = F(f)F(g)$. This is usually called a **group homomorphism** *from G to H.*

2. Suppose G and H are groups regarded as categories, and let $F, F'\colon G \to H$ be a pair of functors. Figure out what a natural transformation $\alpha\colon F \Rightarrow F'$ amounts to.

α consists of an element α of H, such that for any element f of G, $F(f)\alpha = \alpha F'(f)$.

3. Suppose G is a group regarded as a category and let $1_G\colon G \to G$ be the identity functor. Figure out what a natural transformation $\alpha\colon 1_G \Rightarrow 1_G$ amounts to. What is the set of all such natural transformations usually called?

α consists of an element α of G, such that for any element f of G, $f\alpha = \alpha f$. The set of such α is usually called the **centre** *(or* **center***) of G.*

4. Let Vect be the category of vector spaces over your favorite field, where the morphisms are linear transformations. Suppose G is a group regarded as a category. Figure out what a functor $F\colon G \to \mathrm{Vect}$ amounts to. What is such a functor usually called?

F consists of an object F of Vect and a function $F\colon G \to \hom(F, F)$, such that $F(1_G) = 1_F$ and for any pair of elements f, g of G, $F(fg) = F(f)F(g)$. This is usually called a **linear representation** *of G.*

5. Suppose G is a group regarded as a category and let $F, F'\colon G \to \mathrm{Vect}$ be functors. Figure out what a natural transformation $\alpha\colon F \Rightarrow F'$ amounts to. What is such a natural transformation usually

called?

α consists of a linear operator $\alpha\colon F \to F'$, such that for any element f of G, this diagram commutes:

$$\begin{array}{ccc} F & \xrightarrow{F(f)} & F \\ \alpha \downarrow & & \downarrow \alpha \\ F' & \xrightarrow[F'(f)]{} & F' \end{array}$$

α is usually called an **intertwining operator** *from F to F'.*

6. Suppose G is a Lie group, regarded as a one-object category where the morphisms form a manifold. Let $\mathrm{Aut}(G)$ be the category whose objects are smooth invertible functors $F\colon G \to G$ and whose morphisms are smooth invertible natural transformations $\alpha\colon F \to F'$. The objects of $\mathrm{Aut}(G)$ form a Lie group. Any object F in $\mathrm{Aut}(G)$ gives a subset $[F]$ consisting all objects that are isomorphic to it. What do these subsets look like for $G = \mathrm{SO}(3)$? How about for $G = \mathrm{SU}(2)$?

First, let me state some facts about automorphisms of Lie groups. To begin with, every element of G defines an automorphism of G by conjugation. This defines a group homomorphism $G \to \mathrm{Ob}(\mathrm{Aut}(G))$; let $\mathrm{Inn}(G)$ be the image of this homomorphism. Then $\mathrm{Inn}(G)$ is normal; let $\mathrm{Out}(G)$ be the quotient group. If G is a connected compact real form of a simple Lie group, then $\mathrm{Out}(G)$ is isomorphic to the symmetry group of the Dynkin diagram of G. Also, $\mathrm{Inn}(G)$ is isomorphic to G/Z, where Z is the centre of G.

Now, both $\mathrm{SO}(3)$ and $\mathrm{SU}(2)$ are connected compact real forms of the Lie group A_1:

$$\bullet$$

The symmetry group of this Dynkin diagram is trivial, so $\mathrm{Ob}(\mathrm{Aut}(G)) = \mathrm{Inn}(G)$ in both cases. Since $Z(\mathrm{SO}(3))$ is trivial, I have $\mathrm{Ob}(\mathrm{Aut}(G)) = \mathrm{Inn}(G) \cong \mathrm{SO}(3)$ in both cases.

So, suppose F and F' are elements of G, and let α be $F(F')^{-1}$. Then

$$F(f)\alpha = FfF^{-1}F(F')^{-1} = Ff(F')^{-1} = F(F')^{-1}F'f(F')^{-1} = \alpha F'(f).$$

Thus, $F \cong F'$ in $\mathrm{Aut}(G)$. Therefore, there is a unique isomorphism class $[F]$, which is all of $\mathrm{Ob}(\mathrm{Aut}(G)) \cong \mathrm{SO}(3)$.