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Quantum Gravity Seminar
HW #3 — *Connection as a Functor*

1. Let $A\colon [a,b] \to \mathrm{End}(\mathbb{R}^n)$ be a smooth function. For any object $t \in K[a,b]$ let $F(t)$ be the vector space $\mathbb{R}^n$. If $t_0 \to t_1$ is the unique morphism between objects $t_0, t_1 \in K[a,b]$, let $F(t_0 \to t_1)\colon \mathbb{R}^n \to \mathbb{R}^n$ be the map sending any $\psi_0 \in \mathbb{R}^n$ to $\psi(t_1)$, where $\psi\colon [a,b] \to \mathbb{R}^n$ is the unique smooth solution of the initial value problem (henceforth 'the IVP')

$$\frac{d\psi(t)}{dt} = A(t)\psi(t) \qquad \psi(t_0) = \psi_0.$$

We show first that this actually defines a linear map, so that $F(t_0 \to t_1) \in \mathrm{End}(\mathbb{R}^n)$. Let $a\psi_0 + b\phi_0$ be a linear combination of vectors in $\mathbb{R}^n$. Then by definition of F

$$F(t_0 \to t_1)(a\psi_0 + b\phi_0) = \eta(t_1)$$

where $\eta(t)$ satisfies $\dot{\eta}(t) = A(t)\eta(t)$ and $\eta(t_0) = a\psi_0 + b\phi_0$. But since the differential equation is linear, $a\psi + b\phi$ satisfies this IVP (where ψ and ϕ are solutions of the IVP with $\psi(0) = \psi_0$ and $\phi(0) = \phi_0$, respectively) so by the uniqueness theorem (Theorem 1) we have $\eta = a\psi + b\phi$. Thus,

$$F(t_0 \to t_1)(a\psi_0 + b\phi_0) = a\phi(t_1) + b\phi(t_1) = aF(t_0 \to t_1)(\psi_0) + bF(t_0 \to t_1)(\phi_0).$$

So we have an explicit, well-defined formula for $F : K[a,b] \to \mathrm{End}(\mathbb{R}^n)$ as an object map and a morphism map, and we have only to prove that F is a functor. Given composable morphisms $t_0 \to t_1$ and $t_1 \to t_2$ in $K[a,b]$, their composite must be the unique morphism $t_0 \to t_2$. Then for any $\psi_0 \in \mathbb{R}^n$,

$$F(t_0 \to t_2)(\psi_0) = \psi(t_2)$$

where $\psi\colon [a,b] \to \mathbb{R}^n$ is the unique solution of the IVP with $\psi(t_0) = \psi_0$. On the other hand

$$F(t_1 \to t_2)[F(t_0 \to t_1)(\psi_0)] = \eta(t_2)$$

where $\eta(t)$ satisfies $\dot{\eta}(t) = A(t)\eta(t)$ and $\eta(t_1) = F(t_0 \to t_1)(\psi_0) = \psi(t_1)$. By the uniqueness theorem again, this implies $\eta = \psi$. Therefore

$$F(t_1 \to t_2)[F(t_0 \to t_1)(\psi_0)] = \psi(t_2) = F(t_0 \to t_2)(\psi_0)$$

or, in the smart composition order:

$$F(t_0 \to t_1)F(t_1 \to t_2) = F(t_0 \to t_2).$$

Finally, given any object $t \in K[a,b]$, its identity morphism is just the unique morphism $t \to t$. Then $F(t \to t)(\psi_0) = \psi(t)$ where $\psi(t) = \psi_0$. So, $F(1_t) = 1_{\mathbb{R}^n}$, and F is a functor.

2. Let A be a smooth End$(\mathbb{R}^n)$-valued 1-form on the manifold X. We construct a functor

$$F\colon \mathcal{P}(X) \to \mathrm{Vect}$$

as follows. For each $x \in X$ (an object in $\mathcal{P}(X)$) let $F(x) = \mathbb{R}^n$. For any piecewise-smooth path $\gamma\colon [0,T] \to X$ (morphism in $\mathcal{P}(X)$), let $F(\gamma)$ be the linear transformation

$$\psi_0 \mapsto \psi(T)$$

where $\psi\colon [0,T] \to \mathbb{R}^n$ is the unique solution of the equation

$$\frac{d\psi(t)}{dt} = A_{\gamma(t)}(\gamma'(t))\ \psi(t) \tag{1}$$

with $\psi(0) = \psi_0$. The first thing we might worry about is that since γ is only *piecewise* smooth, $\gamma'(t)$ may not be defined for all $t \in [0,T]$. But this causes no problem: we simply pick a partition $0 = t_0 < t_1 < \cdots < t_k = T$ of $[0,T]$ such that $\gamma'(t)$, and hence the above equation, are defined on each subinterval $[t_i, t_{i+1}]$. Then the "unique solution" referred to above is pieced together together from the unique solutions $\psi^{(i)}(t)$ on each subinterval $[t_i, t_{i+1}]$ subject to the conditions $\psi^{(0)}(0) = \psi_0$, $\psi^{(1)}(t_1) = \psi^{(0)}(t_1)$, $\ldots$, $\psi^{(k-1)}(t_{k-1}) = \psi^{(k-2)}(t_{k-1})$. The first of these is the original initial condition $\psi(0) = 0$ and the rest just force

$$\psi(t) := \psi^{(i)}(t) \quad \text{if } t \in [t_i, t_{i+1}]$$

to be continuous.

The proof that $F(\gamma)$ is a linear map is the same as in the previous problem, and uniqueness is obvious, so let's just show that F is a functor.

If we have two composable paths $\gamma_1\colon [0,T_1] \to X$ and $\gamma_2\colon [0,T_2] \to X$ ($\gamma_1(T_1) = \gamma_2(0)$), then their composite is $\gamma\colon [0, T_1 + T_2] \to X$ given by

$$\gamma = \gamma_1\gamma_2 := \begin{cases} \gamma_1(t) & 0 \le t \le T_1 \\ \gamma_2(t - T_1) & T_1 \le t \le T_1 + T_2 \end{cases}$$

Then $F(\gamma)(\psi_0) = \psi(T_1 + T_2)$ were $\psi(0) = \psi_0$ and $\psi(t)$ satisfies Equation 1, which we can rewrite as

$$\frac{d\psi(t)}{dt} = \begin{cases} A_{\gamma_1(t)}(\gamma_1'(t))\ \psi(t) & 0 \le t \le T_1 \\ A_{\gamma_2(t-T_1)}(\gamma_2'(t - T_1))\ \psi(t - T_1) & T_1 \le t \le T_1 + T_2 \end{cases}$$

In particular, we see from this equation that

$$\begin{aligned} F(\gamma)(\psi_0) : &= \psi(T_1 + T_2) \\ &= \psi_2(T_2) \\ &= F(\gamma_2)(\psi_1(T_1)) \\ &= F(\gamma_2)(F(\gamma_1)(\psi_0)). \end{aligned}$$

where ψ_2 is the solution to Equation 1 satisfying $\psi_2(0) = \psi_1(T_1)$, and ψ_1 is the solution to Equation 1 satisfying $\psi_1(0) = \psi_0$. Since ψ_0 was arbitrary, we can rewrite this result (using the smart composition order) as

$$F(\gamma_1\gamma_2) = F(\gamma_1)F(\gamma_2)$$

Finally, if $1_x\colon [0,0] \to X$ is the identity path at some point $x \in X$, then although it is hard to make sense of a differential equation on the interval $[0,0]$, we still have $F(1_x)\psi_0 = \psi(0)$ where ψ must satisfy $\psi(0) = \psi_0$. So, F preserves identities as well, and is therefore a functor.