# Connections as Functors
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1. Define $F : K[a,b] \to Vect$ as follows:

If $c$ is an object in $K[a,b]$, i.e., $c \in [a,b]$,

$$F : c \mapsto \mathbb{R}^n$$

If $f : t_0 \to t_1$ is a morphism in $K[a,b]$,

$$F(f) : v \mapsto \psi(t_1),$$

where $\psi : [a,b] \to \mathbb{R}^n$ is the unique solution of

$$\frac{d\psi(t)}{dt} = A(t)\psi(t) \tag{1}$$

satisfying the condition $psi(t_0) = v$.

Is $F$ a functor?
First we will show that $F(f)$ is a linear map.

Let $f : t_0 \to t_1$; $v_1, v_2 \in \mathbb{R}^n$.
Then $F(f)(v_1 + v_2) = \phi(t_1)$, where $\phi(t_1)$ is the unique solution to (1) such that $\phi(t_0) = v_1 + v_2$.

Furthermore, $F(f)(v_1) = \psi_1(t_1); F(f)(v_1) = \psi_2(t_1)$; where $\psi_1, \psi_2$ are the unique solutions to (1) satisfying $\psi_1(t_0) = v_1$, $\psi_2(t_0) = v_2$.

Now, note that $\psi_1 + \psi_2$ is a solution to (1) as well, and $(\psi_1 + \psi_2)(t_0) = v_1 + v_2 = \phi(t_0)$. Therefore, by uniqueness we must have $\psi_1 + \psi_2 = \phi$, and so

$$\begin{aligned} F(f)(v_1 + v_2) &= \phi(t_1) \\ &= (\psi_1 + \psi_2)(t_1) \\ &= \psi_1(t_1) + \psi_2(t_1) \\ &= F(f)(v_1) + F(f)(v_2). \end{aligned}$$

Now, if $r \in \mathbb{R}$, $F(f)(rv_1) = \phi_r(t_1)$, $\phi_r$ the unique solution to (1) satisfying $\phi_r(t_0) = rv_1$.

But $r\psi_1$ is a solution to (1) with $r\psi_1(t_0) = rv_1$, so again by uniqueness we must have $r\psi_1 = \phi_r$, and so

$$F(f)(rv_1) = \phi_r(t_1) = r\psi_1(t_1) = rF(f)(v_1).$$

Therefore $F(f)$ is a linear map.

Now let $f : t_0 \to t_1, g : t_1 \to t_2$ be composable morphisms in $K[a,b]$ (so $gf : t_0 \to t_2$); let $v \in \mathbb{R}^n$.

Then

$$\begin{aligned} F(f) &: v \mapsto \phi_v(t_1), \\ F(g) &: \phi_v(t_1) \mapsto \psi_v(t_2), \\ F(gf) &: v \mapsto \lambda_v(t_2); \end{aligned}$$

where $\phi_v, \psi_v$, and $\lambda_v$ are the unique solutions to (1) satisfying $\phi_v(t_0) = v$, $\psi_v(t_1) = \phi_v(t_1)$, and $\lambda_v(t_0) = v$.

By uniqueness, we must have $\phi_v = \lambda_v$. Also note that $\phi_v$ is the unique solution to (1) satisfying $\phi_v(t_1) = \psi_v(t_1)$, and so again by uniqueness we must have $\phi_v = \psi_v$, and therefore

$$F(gf)(v) = \lambda_v(t_2) = \phi_v(t_2) = \psi_v(t_2) = F(g)F(f)(v).$$

Therefore $F(gf) = F(g)F(f)$.

Now, is $F(1_{K[a,b]}) = 1_{Vect}$? We must show, for any object $t_0 \in [a,b], 1_{t_0}$ the identity morphism of $t_0$, that $F(1_{t_0}) =$ identity transformation of $\mathbb{R}^n$.

So let $v \in \mathbb{R}^n$; then $F(1_{t_0})(v) = \phi_v(t_0)$, where $\phi_v : [a,b] \to \mathbb{R}^n$ is the unique solution to (1) satisfying $\phi_v(t_0) = v$. But then $F(1_{t_0})(v) = \phi_v(t_0) = v$, so $F(1_{t_0})$ is the identity transformation of $\mathbb{R}^n$.

This completes the check that $F$ is a functor.

It's straightforward to see that there is only only one functor with the desired properties.

3. Define $F : P(X) \to Vect$ as follows:

For any object $x \in P(X)$, $F : x \mapsto \mathbb{R}^n$

For any piecewise smooth path $\gamma : [0,T] \to X$,

$F(\gamma) : \mathbb{R}^n \to \mathbb{R}^n$, defined by

$$F(\gamma) : v \mapsto \psi(T),$$

where $\psi : [0,T] \to \mathbb{R}^n$ is the unique solution to

$$\frac{d\psi(t)}{dt} = A_{\gamma(t)}(\gamma'(t))\psi(t) \tag{2}$$

satisfying the condition $\psi(0) = v$.

Is $F$ a functor?
First we will show that $F(\gamma)$ is a linear map.

Let $\gamma : [0,T] \to X$; $v, w \in \mathbb{R}^n$.
Then

$$\begin{aligned} F(\gamma) &: v \mapsto \psi_v(T), \\ F(\gamma) &: w \mapsto \psi_w(T), \\ F(\gamma) &: v + w \mapsto \psi_{v+w}(T); \end{aligned}$$

where $\psi_v, \psi_w$, and $\psi_{v+w}$ are the unique solutions to (2) satisfying $\psi_v(0) = v, \psi_w(0) = 2$, and $\psi_{v+w}(0) = v + w$.

We must show now that $\psi_{v+w}(T) = \psi_v(T) + \psi_w(T)$. But $\psi_{v+w}$, $\psi_v + \psi_w$ are both solutions to (2) satisfying

$$\psi_{v+w}(0) = v + w = \psi_v(0) + \psi_w(0),$$

and so by uniqueness we must have $\psi_{v+w} = \psi_v + \psi_w$.
The proof that $F(\gamma)(rv) = rF(\gamma)(v)$ for $r \in \mathbb{R}$ is similar to that used for (1.) above.

Now let $\gamma_1, \gamma_2$ be composable morphisms in $P(x)$; $\gamma_1 : [0, T_1] \to X$, $\gamma_2 : [0, T_2] \to X$, such that $\gamma_1(T_1) = \gamma_2(0)$.
Then $(\gamma_1\gamma_2) : [0, T_1 + T_2] \to X$;

$$(\gamma_1\gamma_2)(t) = \begin{cases} \gamma_1(t), & \text{if } 0 \leq t \leq T_1; \\ \gamma_2(t - T_1), & \text{if } T_1 \leq t \leq (T_1 + T_2). \end{cases}$$

Let $v \in \mathbb{R}^n$. Then

$$\begin{aligned} F(\gamma_1\gamma_2) &: v \mapsto \phi_v(T_1 + T_2), \\ F(\gamma_1) &: v \mapsto \psi_v(T_1), \\ F(\gamma_2) &: \psi_v(T_1) \mapsto \lambda_v(T_2); \end{aligned}$$

where $\phi_v : [0, T_1 + T_2] \to \mathbb{R}^n$, $\psi_v : [0, T_1] \to \mathbb{R}^n$, and $\lambda_v : [0, T_2] \to \mathbb{R}^n$ are the unique solutions to (2) satisfying $\phi_v(0) = v$, $\psi_v(0) = v$, and $\lambda_v(0) = \psi_v(T_1)$.

Is $\lambda_v(T_2) = \phi_v(T_1 + T_2)$? This would show $F(\gamma_1\gamma_2) = F(\gamma_2)F(\gamma_1)$.

*Here I'm stuck. I would like to continue by defining a new path* $\widetilde{\psi} : [0, T_1+T_2] \to \mathbb{R}^n$,

$$\widetilde{\psi}(t) = \begin{cases} \psi_v(t), & \text{if } 0 \le t \le T_1; \\ \lambda_v(t - T_1), & \text{if } T_1 \le t \le (T_1 + T_2) \end{cases}$$

*and then show that* $\widetilde{\psi}$ *is a solution to* (2), *with* $\widetilde{\psi}(0) = v$. *I could then conclude that* $\widetilde{\psi} = \phi_v$, *and therefore*

$$\lambda_v(T_2) = \widetilde{\psi}(T_1 + T_2) = \phi_v(T_1 + T_2),$$

*which would show* $F(\gamma_1\gamma_2) = F(\gamma_2)F(\gamma_1)$.

*But I can't show* $\widetilde{\psi}$ *is a solution to* (2), *because I have no guarantee that* $\widetilde{\psi}$ *is even differentiable at*$T_1$.

**Actually, it's my fault for being sloppy in stating the problem! The composite $\gamma_1\gamma_2$ of two smooth paths need not be differentiable at the time where one path ends and the next one starts, so literally speaking, the differential equation (2) makes no sense at that value of $t$. More generally, the problem is that our paths are only piecewise smooth, so their derivative can fail to exist at finitely many values of $t$. To get around this, we can do various things. First, we can demand that (2) hold only at those times for which the derivative of the path $\gamma$ exists, and demand that $\psi(t)$ be continuous at all times: this is enough to get a unique solution $\psi(t)$ pieced together out of solutions defined on the intervals where $\gamma$ is smooth. Or, second, we can replace the differential equation (2) by the corresponding integral equation**

$$\psi(t) = \psi_0 + \int_0^t A_{\gamma(s)}(\gamma'(s))\, \psi(t) ds.$$

**This makes sense even when $\gamma(s)$ is just piecewise smooth, since the integrand will be bounded, and continuous except at finitely many points. — John Baez**