Duals

John C. Baez, November 4, 2004

The concept of 'dual vector space' has a massive generalization in terms of category theory. It goes like this....

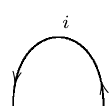
Suppose C is a monoidal category. An **adjunction** *in C is a quadruple (x, x^*, i, e) where:*

- *x and x^* are objects in C.*
- *$i\colon 1 \to x \otimes x^*$ and $e\colon x^* \otimes x \to 1$ are morphisms in C (called the* **unit** *and* **counit** *of the adjunction, respectively).*
- *The following diagrams commute:*

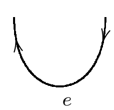
$$\begin{array}{ccccc} 1\otimes x & \xrightarrow{i\otimes 1} & (x\otimes x^*)\otimes x & \xrightarrow{a_{x,x^*,x}} & x\otimes(x^*\otimes x) \\ \downarrow{\scriptstyle \ell_x} & & & & \downarrow{\scriptstyle 1\otimes e} \\ x & & \xrightarrow[r_x^{-1}]{} & & x\otimes 1 \end{array}$$

$$\begin{array}{ccccc} x^*\otimes 1 & \xrightarrow{1\otimes i} & x^*\otimes(x\otimes x^*) & \xrightarrow{a^{-1}_{x^*,x,x^*}} & (x^*\otimes x)\otimes x^* \\ \downarrow{\scriptstyle r_{x^*}} & & & & \downarrow{\scriptstyle e\otimes 1} \\ x^* & & \xrightarrow[\ell_{x^*}^{-1}]{} & & 1\otimes x^* \end{array}$$

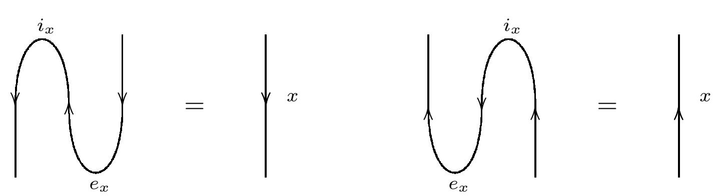
Aaron Lauda has dubbed the above commutative diagrams the **zig-zag identities**. *Why? The string diagram for the unit $i\colon 1 \to x \otimes x^*$ looks like this:*



where it is understood that the downward pointing arrow corresponds to x and the upward pointing arrow to x^*. Similarly, the counit $e\colon x^* \otimes x \to 1$ looks like this:



These string diagrams are reminiscent of the Feynman diagrams for the creation and annihilation of particle/antiparticle pairs! In this notation, the zig-zag identities simply say that we can straighten a zig-zag in a piece of string:



1. The category Vect_k has finite-dimensional vector spaces over a fixed field k as its objects and linear maps between these as its morphisms. Vect becomes a monoidal category with the usual tensor product of vector spaces and with the unit object $1 = k$.

Suppose $V \in \mathrm{Vect}_k$ and V^* is its dual, i.e. the space of all linear maps $f\colon V \to k$. Define $i_V\colon k \to V \otimes V^* \cong \mathrm{End}(V)$ by

$$i_V(\alpha) = \alpha\, 1_V$$

and define $e_V\colon V^* \otimes V \to k$ by

$$e_V(f \otimes v) = f(v).$$

a. Show that (V, V^*, i_V, e_V) is an adjunction.

b. What goes wrong when V is infinite-dimensional?

In the above situation we often call V^* 'the' dual of V, but one should be a bit careful. After all, the precise definition of 'linear map' depends on the definition of 'function', and different people use slightly different definitions of 'function' — for example, by saying a function is a set of ordered pairs, but using different definitions of 'ordered pair', such as Norbert Wiener's original 1914 definition $(x, y) = \{\{\{x\}, \emptyset\}, \{\{y\}\}\}$, Kazimierz Kuratowski's more efficient 1921 definition $(x, y) = \{\{x\}, \{x, y\}\}$, or his brother Zreimizak's 1922 definition $(x, y) = \{\{y\}, \{y, x\}\}$. (Tragically, Kazimierz and Zreimizak killed each other in a foolish swordfight over this issue in 1923.)

So, if we were being incredibly nitpicky, we might call V^* 'a' dual of V. The concept of adjunction makes this more precise, by saying exactly what a dual should be like — at least in the finite-dimensional case. And the really nice thing is that we can prove that any two duals of the same object are isomorphic in a god-given way:

2. Suppose x is an object in the monoidal category C and (x, y, i, e) and (x, y', i', e') are adjunctions.

a. Construct an isomorphism $f\colon y \to y'$.

b. Describe the sense in which the isomorphism $f\colon y \to y'$ makes (x, y, i, e) and (x, y', i', e') into isomorphic adjunctions.

(Hint: it's easiest to do these using string diagrams.)

This result means we're allowed to speak of 'the dual' of x as long as we use the word 'the' in its official category-theoretic sense. In set theory, we're allowed to speak of **the** *element with some property whenever such an object exists and any two elements with this property are equal. In category theory, we're allowed to speak of* **the** *object equipped with some stuff whenever such an object exists and any two objects equipped with this stuff are isomorphic in a specified way.*

Finally, let's show that monoidal functors automatically preserve duals of objects:

3. Suppose C and D are monoidal categories and $F\colon C \to D$ is a monoidal functor. Show that if (x, y, i, e) is an adjunction in C, there is an adjunction in D making $F(y)$ into the dual of $F(x)$.

(Hint: when F is a strict monoidal functor this adjunction in D is just $(F(x), F(y), F(i), F(e))$, but in general we need to keep track of the fact that F preserves the tensor product and unit object only up to specified isomorphisms.)