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QUANTUM GRAVITY SEMINAR
HW # 4 — Duals

1.) Let $Vect_k$ be the monoidal category of finite dimensional vector spaces over k. The canonical isomorphism $V \otimes V^* \cong End(V)$ is given by

$$\begin{aligned} h : V \otimes V^* &\longrightarrow End(V) \\ v \otimes f &\longmapsto f(-)v. \end{aligned}$$

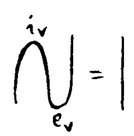
If $e_1, e_2, \ldots, e_n$ is a basis for V with $e^1, e^2, \ldots, e^n$ the dual basis for V^*, then under the isomorphism h, $1_V \in End(V)$ can be written as $e_i \otimes e^i$, where we use Einstein summation convention. For, if $w = w^i e_i \in V$, then

$$\begin{aligned} h(e_i \otimes e^i)(w) &= e^i(w^j e_j) e_i \\ &= w^j \delta^i_j e_i \\ &= w^i e_i \\ &= w. \end{aligned}$$

So we define $i_V : k \to V \otimes V^*$ by $i_V(\alpha) = \alpha e_i \otimes e^i$
and $e_V : V^* \otimes V \to k$ by $e_V(f \otimes v) = f(v)$.

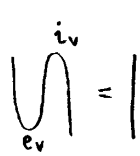
To see that (V, V^*, i_V, e_V) is an adjunction: $\forall \alpha \in k$, $v \in V$, $f \in V^*$ we have:

$$\begin{tikzcd}
\alpha \otimes v \arrow[r, mapsto, "i \otimes 1"] \arrow[ddd, mapsto, "l_V"'] & \alpha(e_i \otimes e^i) \otimes v \arrow[r, mapsto, "\text{assoc.}"] & \alpha e_i \otimes (e^i \otimes v) \arrow[d, mapsto, "1 \otimes e_V"] \\
 & & \alpha e_i \otimes v^i \quad \text{— where } v = v^i e_i \quad v^i \in k \arrow[d, equal] \\
 & & \alpha v^i e_i \otimes 1_k \arrow[d, equal] \\
\alpha v \arrow[rr, "r_V^{-1}"] & & \alpha v \otimes 1
\end{tikzcd}$$



and

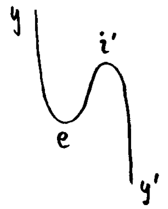
$$\begin{tikzcd}
f \otimes \alpha \arrow[r, mapsto, "1 \otimes i_V"] \arrow[dddd, mapsto, "r_{V^*}"'] & f \otimes (\alpha e_i \otimes e^i) \arrow[r, mapsto, "\text{assoc.}^{-1}"] & (f \otimes \alpha e_i) \otimes e^i \arrow[d, mapsto, "e_V \otimes 1"] \\
 & & f(\alpha e_i) \otimes e^i \arrow[d, equal] \\
 & & \alpha f_i \otimes e^i \quad \text{— where } f = f_i e^i \quad f_i \in k \arrow[d, equal] \\
 & & 1_k \otimes \alpha f_i e^i \arrow[d, equal] \\
\alpha f \arrow[rr, "l_{V^*}^{-1}"] & & 1_k \otimes \alpha f
\end{tikzcd}$$



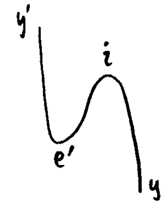
(b) Note: if V is not finite dimensional, then we might not have $V \cong V^*$, so $End(V) \ncong V \otimes V^*$.

2.) Let (x,y,i,e) and (x,y',i',e') be adjunctions in the monoidal category C (so in particular, y & y' are both "dual" to x)

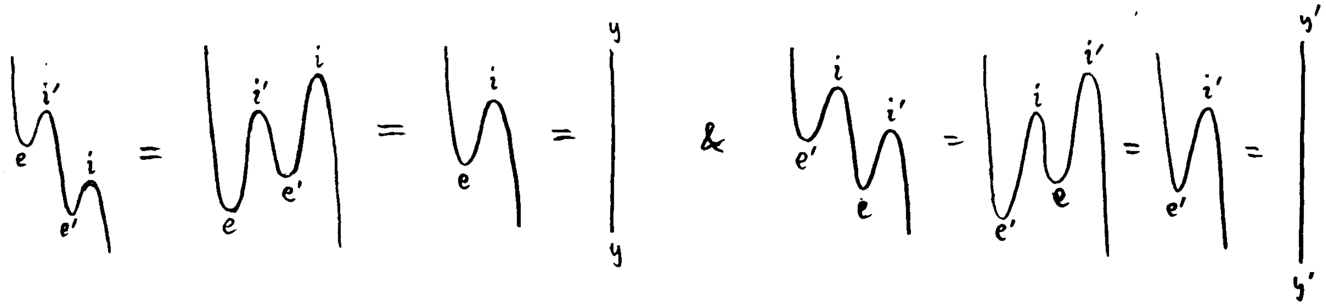
(a) Using string diagrams, we have an obvious morphism from y to y':



which has inverse



, since

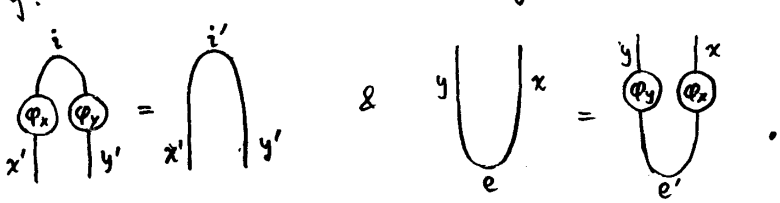


(b) Given a monoidal category C, there is a category whose objects are adjunctions (x,y,i,e) in C and whose morphisms are pairs of morphisms in C making the obvious diagrams commute. Namely, $\varphi \in \hom((x,y,i,e),(x',y',i',e'))$ consists of morphisms $\varphi_x : x \longrightarrow x'$ and $\varphi_y : y \longrightarrow y'$ in C such that

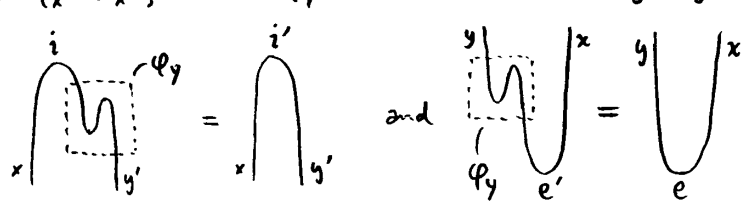
$$\begin{tikzcd} & 1 \arrow[dl, "i"'] \arrow[dr, "i'"] & \\ x\otimes y \arrow[rr, "\varphi_x\otimes\varphi_y"'] & & x'\otimes y' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} y\otimes x \arrow[rr, "\varphi_y\otimes\varphi_x"] \arrow[dr, "e"'] & & y'\otimes x' \arrow[dl, "e'"] \\ & 1 & \end{tikzcd}$$

commute.

So the natural meaning of isomorphic adjunctions is just an isomorphism in this category. The above commutative diagrams can be rewritten as string diagrams:



To see that any two adjunctions (x,y,i,e) and (x,y',i',e') are isomorphic, we let $\varphi_x = 1_x$, and let φ_y be the isomorphism $y \longrightarrow y'$ constructed in (a). Then



by the zigzag identities.

(This shows any two "duals of x" are canonically isomorphic)

3) Our monoidal functor $F: C \to D$ comes equipped with a natural isomorphism

$$\Phi : F(-) \otimes F(-) \Longrightarrow F(- \otimes -)$$

between the functors

$$F(-) \otimes F(-) : C \times C \longrightarrow C'$$
$$\& \quad F(- \otimes -) : C \times C \longrightarrow C',$$

and an isomorphism

$$\phi : 1_D \longrightarrow F(1_C)$$

making certain diagrams commute. (To avoid drawing those here, I'm using the notation of the handout "Some Definitions Everyone Should Know").

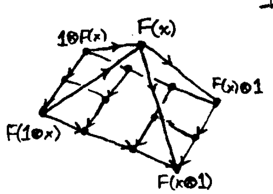
Now given that (x, y, i, e) is an adjunction in C, we wish to construct an adjunction $(F(x), F(y), \iota, \varepsilon)$ in D. With the given data, there's really only one way to sensibly define ι & ε: Let ι & ε be defined by declaring the following diagrams commutative:

$$\begin{tikzcd}
1_D \arrow[r, dashed, "\iota"] \arrow[d, "\phi"', "\wr"] & F(x) \otimes F(y) \\
F(1_C) \arrow[r, "F(i)"'] & F(x \otimes y) \arrow[u, "\wr"', "\Phi^{-1}_{x,y}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F(y) \otimes F(x) \arrow[r, "\varepsilon"] \arrow[d, "\Phi_{y,x}"', "\wr"] & 1_D \\
F(y \otimes x) \arrow[r, dashed, "F(e)"'] & F(1_C) \arrow[u, "\wr"', "\phi^{-1}"]
\end{tikzcd}$$

The proof that this choice makes $(F(x), F(y), \iota, \varepsilon)$ an adjunction is now just a big jigsaw puzzle: there are only so many ways to stick commutative diagrams together.

$$
\begin{tikzcd}[column sep=large]
1_D \otimes F(x) \arrow[r, "i \otimes 1_{F(x)}"] \arrow[d, "\phi \otimes 1_{F(x)}"'] \arrow[dr, phantom, "\text{commutes by definition of } i"] & (F(x) \otimes F(y)) \otimes F(x) \arrow[rr, "a_{F(x),F(y),F(x)}"] \arrow[d, "\Phi_{x,y} \otimes 1_{F(x)}"] & & F(x) \otimes (F(y) \otimes F(x)) \arrow[r, "1_{F(x)} \otimes \varepsilon"] \arrow[d, "1_{F(x)} \otimes \Phi_{y,x}"'] \arrow[dr, phantom, "\text{commutes by definition of } \varepsilon"] & F(x) \otimes 1_D \arrow[d, "1_{F(x)} \otimes \phi"] \\
F(1_C) \otimes F(x) \arrow[r, "F(i) \otimes 1_{F(x)}"] \arrow[d, "\Phi_{1_C,x}"'] \arrow[dr, phantom, "\text{commutes by naturality of } \Phi"] & F(x \otimes y) \otimes F(x) \arrow[d, "\Phi_{x \otimes y,x}"] & \text{commutes since } F \text{ is a monoidal functor} & F(x) \otimes F(y \otimes x) \arrow[r, "1_{F(x)} \otimes F(e)"] \arrow[d, "\Phi_{x,y \otimes x}"'] \arrow[dr, phantom, "\text{commutes by naturality of } \Phi"] & F(x) \otimes F(1_C) \arrow[d, "\Phi_{x,1_C}"] \\
F(1_C \otimes x) \arrow[r, "F(i \otimes 1_x)"] \arrow[ddrr, "F(l_x)"] & F((x \otimes y) \otimes x) \arrow[rr, "F(a_{x,y,x})"] & \text{commutes since } (x,y,i,e) \text{ is an adjunction in } C \text{ (This is just } F \text{ applied to one of the zigzags)} & F(x \otimes (x \otimes y)) \arrow[r, "F(1_x \otimes e)"] & F(x \otimes 1_C) \\
F(1_C) \otimes F(x) \arrow[u, "\Phi_{1_C,x}"] & \text{commutes since } F \text{ is a monoidal functor} & & \text{commutes since } F \text{ is a monoidal functor} & F(x) \otimes F(1_C) \arrow[u, "\Phi_{x,1_C}"'] \\
1_D \otimes F(x) \arrow[u, "\phi \otimes 1_{F(x)}"] \arrow[rr, "l_{F(x)}"'] & & F(x) \arrow[rr, "r^{-1}_{F(x)}"'] \arrow[uurr, "F(r_x^{-1}) = F(r_x)^{-1}"] & & F(x) \otimes 1_D \arrow[u, "1_{F(x)} \otimes \phi"']
\end{tikzcd}
$$

Fold along this edge to get a pyramid:



Since all faces of the pyramid commute except for one, the one must commute as well.

$$
\therefore \quad
\begin{tikzcd}
1_D \otimes F(x) \arrow[r, "i \otimes 1_{F(x)}"] \arrow[d, "l_{F(x)}"'] & (F(x) \otimes F(y)) \otimes F(x) \arrow[r, "a_{F(x),F(y),F(x)}"] & F(x) \otimes (F(y) \otimes F(x)) \arrow[d, "1_{F(x)} \otimes \varepsilon"] \\
F(x) \arrow[rr, "r^{-1}_{F(x)}"'] & & F(x) \otimes 1_D
\end{tikzcd}
\quad \text{commutes.}
$$

Similarly, since every tile in the following diagram commutes:

$$\begin{tikzcd}
F(y)\otimes 1_D \arrow[r, "1_{F(y)}\otimes\iota"] \arrow[d, dashed, "1_{F(y)}\otimes\phi"'] \arrow[dr, phantom, "\text{commutes by definition of }\iota"] & F(y)\otimes(F(x)\otimes F(y)) \arrow[rr, "a^{-1}_{F(y),F(x),F(y)}"] \arrow[d, dashed, "1_{F(y)}\otimes\Phi_{x,y}"] \arrow[drr, phantom, "\text{commutes since } F \text{ is a monoidal functor}"] & & (F(y)\otimes F(x))\otimes F(y) \arrow[r, "\varepsilon\otimes 1_{F(y)}"] \arrow[d, dashed, "\Phi_{y,x}\otimes 1_{F(y)}"'] \arrow[dr, phantom, "\text{commutes by definition of }\varepsilon"] & 1_D\otimes F(y) \arrow[d, "\phi\otimes 1_{F(y)}"] \\
F(y)\otimes F(1_C) \arrow[r, "1_{F(y)}\otimes F(i)"] \arrow[d, dashed, "\Phi_{y,1_C}"'] \arrow[dr, phantom, "\text{commutes by naturality of }\Phi"] & F(y)\otimes F(x\otimes y) \arrow[d, dashed, "\Phi_{y,x\otimes y}"] & & F(y\otimes x)\otimes F(y) \arrow[r, "F(e)\otimes 1_{F(y)}"] \arrow[d, dashed, "\Phi_{y\otimes x,y}"'] \arrow[dr, phantom, "\text{commutes by naturality of }\Phi"] & F(1_C)\otimes F(y) \arrow[d, "\Phi_{1_C,y}"] \\
F(y\otimes 1_C) \arrow[r] \arrow[ddrr, "F(r_y)"] & F(y\otimes(x\otimes y)) \arrow[rr, "F(a_{y,x,y})^{-1}"] \arrow[dr, phantom, "\text{commutes since }(x,y,i,e)\text{ is an adjunction in }C"] & & F((y\otimes x)\otimes y) \arrow[r, "F(e\otimes 1_y)"] & F(1_C\otimes y) \\
F(y)\otimes F(1_C) \arrow[u, dashed, "\Phi_{y,1_C}"] \arrow[dr, phantom, "\text{commutes since } F \text{ is a monoidal functor}"] & & & & F(1_C)\otimes F(y) \arrow[u, dashed, "\Phi_{1_C,y}"'] \\
F(y)\otimes 1_D \arrow[u, dashed, "1_{F(y)}\otimes\phi"] \arrow[rr, "r_{F(x)}"'] & & F(x) \arrow[uurr, "F(l_y)^{-1}"] \arrow[rr, "l^{-1}_{F(y)}"'] & & 1_D\otimes F(y) \arrow[u, dashed, "\phi\otimes 1_{F(y)}"'] \arrow[ul, phantom, "\text{commutes since } F \text{ is a monoidal functor}"]
\end{tikzcd}$$

,

$$\begin{tikzcd}
F(y)\otimes 1_D \arrow[r, "1_{F(y)}\otimes\iota"] \arrow[d, "r_{F(y)}"'] & F(y)\otimes(F(x)\otimes F(y)) \arrow[r, "a^{-1}_{F(y),F(x),F(y)}"] & (F(y)\otimes F(x))\otimes F(y) \arrow[d, dashed, "\varepsilon\otimes 1_{F(y)}"] \\
F(y) \arrow[rr, "l^{-1}_{F(y)}"'] & & 1_D\otimes F(y)
\end{tikzcd}$$

commutes.

$\therefore$ $(F(x), F(y), \iota, \varepsilon)$ is an adjunction. ∎