Quantum Gravity Seminar

"Gauge Theory
&
Topology"

Fall 2004

John C. Baez
(Notes by Derek K. Wise)

© 2004 John Baez & Derek Wise

QUANTUM GRAVITY SEMINAR
Fall 2004

GAUGE THEORY & TOPOLOGY

(or: symmetry, forces, & spacetime)

23 Sept 2004

We'll take a "postmodern" approach not starting with the assumption that spacetime is a manifold. For this, it's best to start with a little history of "n-categorical physics."

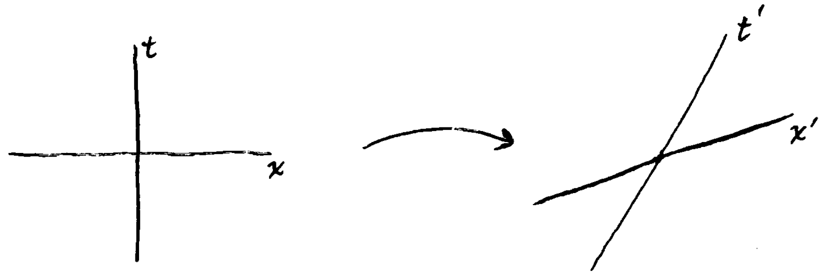
Poincaré (1894) – He invented the fundamental group $\pi_1(X)$ of a space X with basepoint $* \in X$. This turns a <u>space</u> into a <u>group</u>. Later, Eilenberg & MacLane figured out a way to turn a group G into an "Eilenberg-MacLane space" $K(G,1)$ with $\pi_1(K(G,1)) = G$, <u>and</u> $\pi_n(K(G,1)) = \{0\}$ $\forall n \neq 1$. (Uniquely determined – up to homotopy equivalence – by this property!)

Relativity –

Maxwell (1876) – emphasized position & velocity can only be measured <u>relative</u> to other bodies; formulated equations for electromagnetism in which light has the same velocity for everyone!

© John Baez & Derek Wise

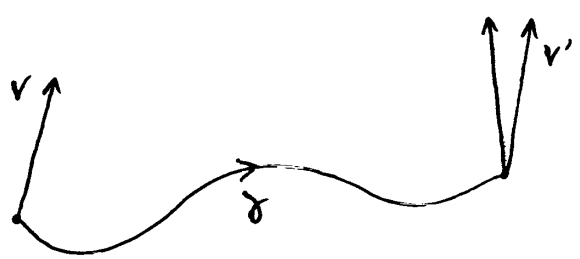
Poincaré (June 1905) invented the Lorentz group – generated by Lorentz boosts



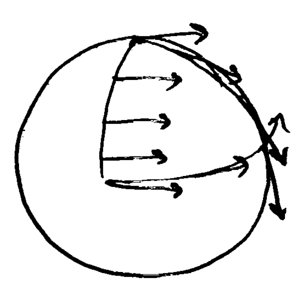
& rotations.

Einstein (June 30, 1905) – first paper on special relativity. He showed that Maxwell's Equations + relativity principle $\Rightarrow$ Lorentz boosts are symmetries of laws of physics.

Einstein (1913) – learned about the Levi-Civita connection & put this to use in his studies of gravity.

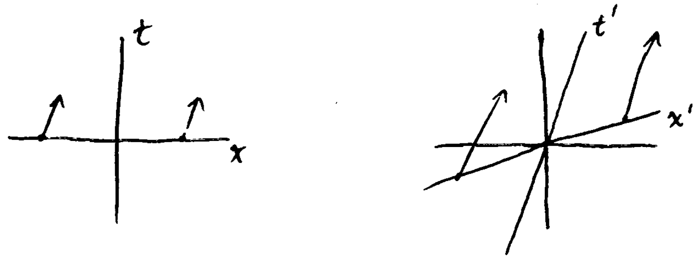


To compare vectors at different points in curved space(time) we need to "parallel transport" one vector to the other along some path, γ. Note this depends on the path chosen:



If γ is a loop in spacetime, parallel transport around γ determines an element L of the Lorentz group: carrying v around the loop γ gives $v' = Lv$. Note: now even the concept of being "at rest relative to another object" becomes ambiguous, since it depends on a choice of path

(In SR:

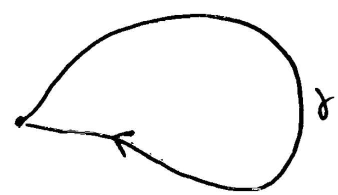


if two vectors were equal in the original frame, then they'll be equal in the transformed frame.)

We're replacing the naive notion of "equality" of velocity vectors by the notion of "isomorphism" - which depends on choice of path.

Einstein (1915) - came up with general relativity: equations describing gravity. For a particle in free fall, its velocity vector is parallel transported along the path it traces out: i.e. it traces out a geodesic.

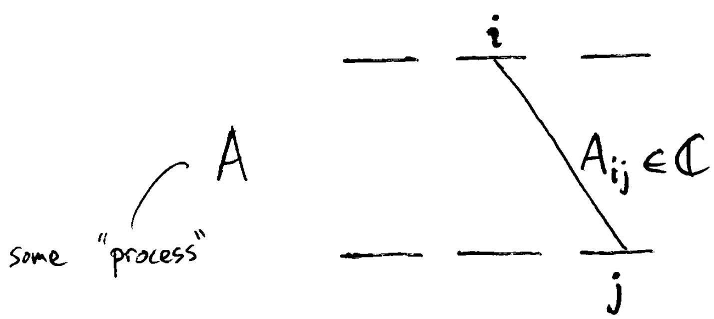
Weyl (1918) — introduced the concept of "gauge invariance" while trying to unify Maxwell's equations of electromagnetism & Einstein's equations of gravity. Weyl suggested that the <u>length</u> of a vector might also change under parallel transport in a path dependent way (this is the origin of the term <u>gauge</u> in physics). Einstein pointed out this can't work, but the idea was on the right track, because electromagnetism really <u>is</u> described by a connection that assigns a number to any loop in spacetime:



but this number describes not dilation of a ruler, but change of phase of a wavefunction. This number is not in $\mathbb{R}^+$ but in $U(1)$.

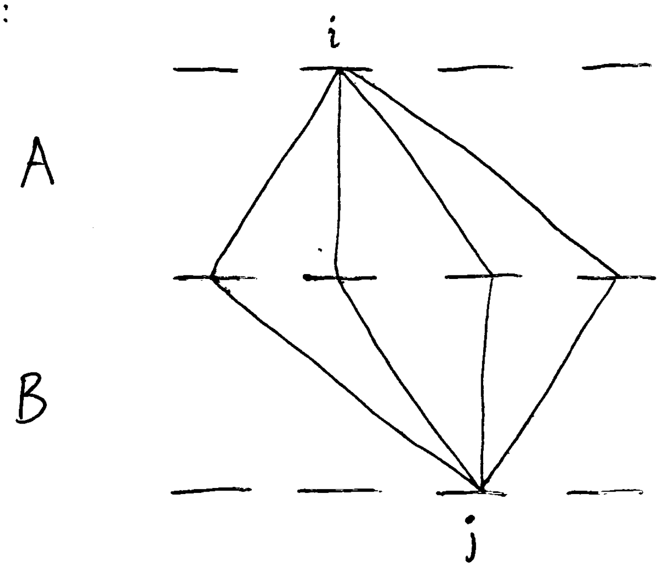
Quantum Mechanics —

Heisenberg (1925) invented quantum mechanics in the form of "matrix mechanics": if a system has a set of states $\{1, 2, \ldots, n\}$, transitions between states:



have "amplitudes": the amplitude to go from state i to state j is a complex number $A_{ij} \in \mathbb{C}$.

This:



$$(AB)_{ij} = \sum_k A_{ik} B_{kj}$$

is how Heisenberg "composed" processes – thus reinventing matrix multiplication