28 Sept 2004

Born (1928) — He gave the "probability interpretation" of Heisenberg's transition amplitudes $A_{ij} \in \mathbb{C}$, namely: $|A_{ij}|^2$ is the probability to go from state $i$ to state $j$ via the process $A$.

Back to the Math side of the story...

Eilenberg, MacLane (1945) — Described categories, functors, & natural transformations. A category is a collection of objects & morphisms between these:

$$x \xrightarrow{f} y$$

where we can "compose" morphisms:

$$x \xrightarrow{f} y \qquad y \longrightarrow z$$

to get

$$x \xrightarrow{fg} z$$

s.t. composition is associative: $(fg)h = f(gh)$ & each object $x$ has $1_x$ s.t. $1_x f = f = f 1_x$.

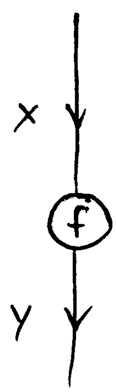
For Eilenberg & MacLane a typical example would be

**Top** : objects are spaces
morphisms are continuous maps

or

**Grp** : objects are groups
morphisms are group homomorphisms

but later, examples of a different sort emerged where $f$ is some sort of "black box" with one input "wire" $x$ & one output "wire" $y$ :



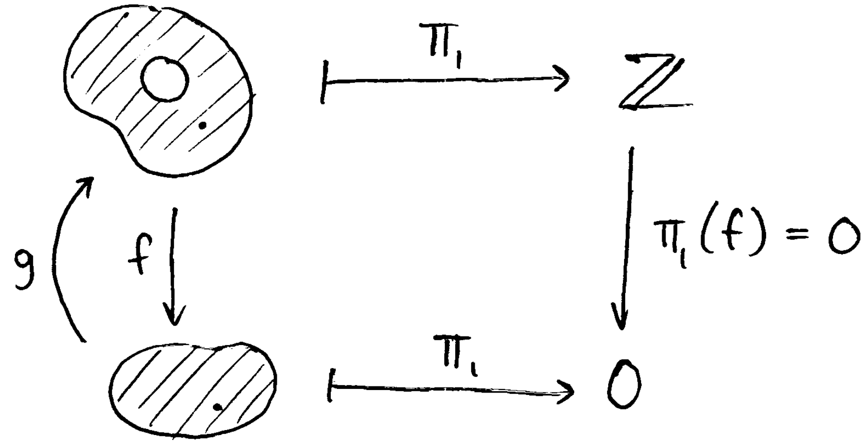

This style of drawing is "Poincaré dual" to the other ($x \xrightarrow{f} y$)

Eilenberg & MacLane were interested in functors going between categories, e.g.

$$\mathrm{Top}_* \xrightarrow{\pi_1} \mathrm{Grp}$$

(pointed spaces)

Example: the "No Retraction Theorem"




Note: this diagram proves $\nexists g$ s.t. $fg = 1$

...

... since if there were:

$$\begin{array}{ccc} \pi_1(fg) & = & \pi_1(1_{\circ}) \\ \| & & \| \\ \pi_1(f)\pi_1(g) & & 1_{\mathbb{Z}} \\ \| & & \\ 0 & & \end{array}$$

contradiction!

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Also they considered natural transformations between functors:

$$\begin{tikzcd} \mathrm{Top}_* \arrow[r, bend left=50, "\pi_1"{name=U}] \arrow[r, bend right=50, "H_1"'{name=D}] & \mathrm{Grp} \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd}$$

where $H_1$ is the first homology.

$$\alpha_X : \pi_1(X) \longrightarrow H_1(X)$$

is "natural" because $f : X \to Y$ gives:

$$\begin{tikzcd} \pi_1(X) \arrow[r, "\alpha_X"] \arrow[d, "\pi_1(f)"'] & H_1(X) \arrow[d, "H_1(f)"] \\ \pi_1(Y) \arrow[r, "\alpha_Y"] & H_1(Y) \end{tikzcd}$$

which <u>commutes</u>.

They also constructed a functor $K(-,1)$:

$$\mathrm{Grp} \xrightarrow{K(-,1)} \mathrm{Top}_* \qquad (G \mapsto K(G,1))$$

with the property that

$$\begin{tikzcd}
\mathrm{Grp} \arrow[r, "K(-{,}1)"] \arrow[rr, bend right=60, "1_{\mathrm{Grp}}"'] & \mathrm{Top}_* \arrow[r, "\pi_1"] \arrow[d, phantom, "\wr\Downarrow\alpha"] & \mathrm{Grp} \\
& {} &
\end{tikzcd}$$

where $\alpha$ is a natural isomorphism, meaning a natural trans. s.t. $\alpha_G : \pi_1(K(G,1)) \longrightarrow G$ is an isomorphism $\forall G$. (But $K(\pi_1(X),1) \not\cong X$, since the other homotopy gps of $X$ have been killed)

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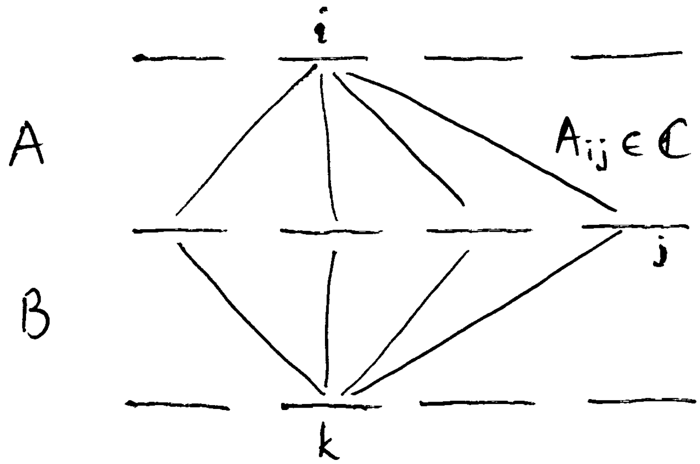
Soon after (and quite uninfluenced by F. & M.)...

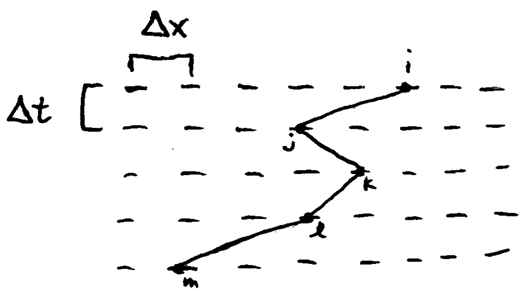
Feynman (1947) – Shelter Island Conference.
He described work on some strange new ideas in quantum mechanics & quantum field theory. First, he invented "Feynman path integrals."

Recall Heisenberg had this picture:




$$(AB)_{ik} = \sum_j A_{ij} B_{jk} .$$

Feynman used this to describe the motion of a particle on a line by discretizing time and space:




4 steps

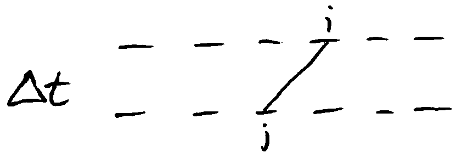
$$(A^4)_{im} = \sum_{j,k,\ell} A_{ij} A_{jk} A_{k\ell} A_{\ell m} (\Delta x)^3$$

(like a triple integral)

and then taking a limit as $\Delta x \to 0$ and $\Delta t \to 0$. He showed that one got Schrödinger's eqn (the right answer) if

$$A_{ij} = e^{\frac{\sqrt{-1}}{\hbar} S_{ij}}$$

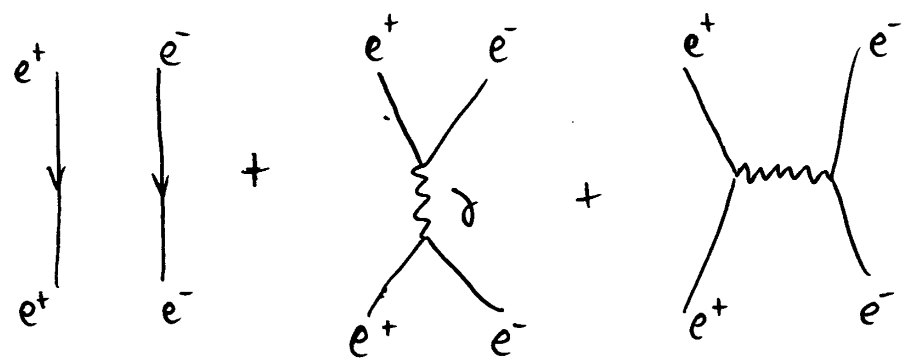
where $S_{ij}$ is the <u>action</u> of this path

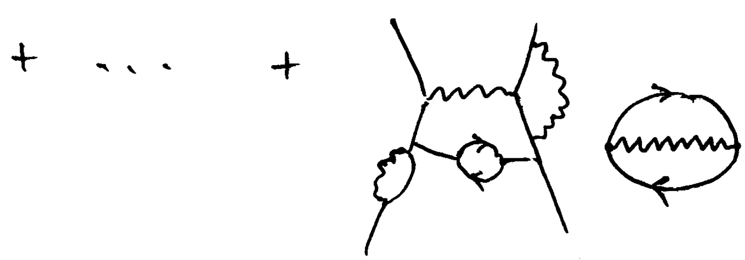


namely:

$$S_{ij} = \int (\text{kinetic energy} - \text{potential energy})\, dt$$

He then applied this path integral technology to quantum field theory, which treats systems of many interacting particles. The upshot was a theory of "Feynman diagrams" in which amplitudes for some bunch of particles to become some other bunch of particles (in certain states) is given as a sum over diagrams:



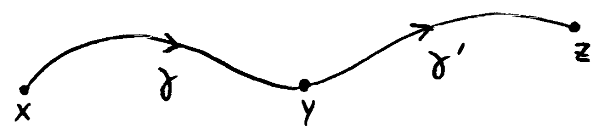

These diagrams started out as a "trick", but later took on a life of their own, which requires some n-category theory to fully understand.

30 Sept. 2004

An Ahistorical Digression — We've seen that in gauge theory, parallel transport describes how things transform as we move them along paths in space(time). Naively, we can say parallel transport assigns a group element $g_\gamma \in G$ to each path $\gamma$, s.t.

$$g_\gamma g_{\gamma'} = g_{\gamma\gamma'}$$

where $\gamma\gamma'$ is the concatenation of the paths $\gamma$ & $\gamma'$:




Also:

$$g_{1_x} = 1_G$$

where $1_x$ is the identity path from x to itself.

So in fact, parallel transport defines a functor from some category whose objects are points of space(time) & morphisms are paths, to G. This category is a relative of Poincaré's "fundamental group," but

1) we're not only considering loops based at some chosen point (so in this way it's more similar to the fundamental groupoid)

2) We're not taking homotopy classes of paths, but only reparameterization classes of paths (to make associative and unit laws hold)

In electromagnetism, $G = U(1)$ & what transforms is the "phase" of a charged particle. Notice that in Feynman's path integrals, the amplitude for a particle to go from $x$ to $y$ is given by an integral over all paths $\gamma$ from $x$ to $y$, where the integrand assigns to each path a phase:

$$e^{iS(\gamma)/\hbar} \in U(1)$$

where $S(\gamma)$ is the action of the path $\gamma$.

In fact, the action is (or "gives") a $U(1)$ connection (i.e. a functor from the category of paths to $U(1)$) on ~~phase space~~ the cotangent bundle of spacetime, while the electromagnetic field gives a $U(1)$ connection on spacetime.

$$\left[\begin{array}{l} t, x, y, z \\ E, p_x, p_y, p_z \end{array}\right.$$

$T^*\mathbb{R}^4$

Back to our history:

MacLane (1963) — Invented monoidal & symmetric monoidal categories. A monoidal category is (roughly) a category equipped with a "tensor product" — e.g:

Vect with usual $\otimes$

Set with $\times$ (Cartesian Product)

Set with $+$ (Disjoint Union)

Note: just as monoids are sets equipped with an associative product & unit, monoidal categories are <u>categories</u> equipped with associative product and unit, where laws hold only up to specified isomorphisms, e.g. the associator

$$a_{x,y,z} : (x \otimes y) \otimes z \longrightarrow x \otimes (y \otimes z).$$

These should be <u>natural</u> isomorphisms and

$$\begin{tikzcd}
& (w\otimes(x\otimes y))\otimes z \arrow[r] & w\otimes((x\otimes y)\otimes z) \arrow[dr] & \\
((w\otimes x)\otimes y)\otimes z \arrow[ur, "a_{w,x,y}\otimes 1_z"] \arrow[dr] & & & w\otimes(x\otimes(y\otimes z)) \\
& (w\otimes x)\otimes(y\otimes z) \arrow[urr, "a_{w,x,y\otimes z}"'] & &
\end{tikzcd}$$

should commute.

MacLane showed that any mon. cat. is equivalent to a strict one, i.e. one where the associators and

$$\ell_x : 1 \otimes x \xrightarrow{\sim} x$$
$$r_x : x \otimes 1 \longrightarrow x$$

are all identity morphisms. This is a version of "MacLane's coherence theorem."

He also wrote down axioms for a "symmetry":

$$S_{x,y} : x \otimes y \longrightarrow y \otimes x$$

in a monoidal category, thus defining a "symmetric monoidal category." These categorify the concept of a commutative monoid.

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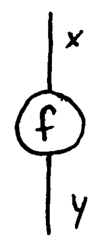
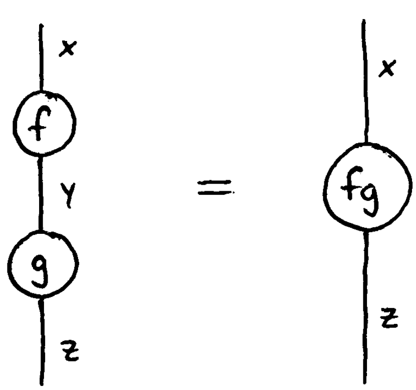
In a category, we can draw morphisms like this




& composition as

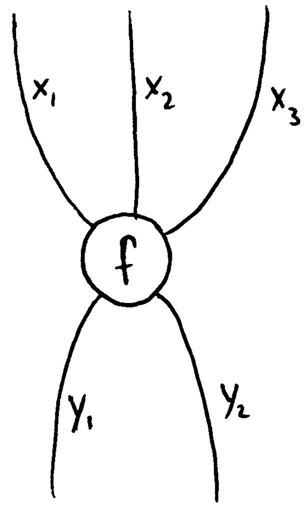



& identity morphisms:

In a (strict) monoidal category we can draw a morphism

$$f: x_1 \otimes x_2 \otimes x_3 \longrightarrow y_1 \otimes y_2$$

as




which we can compose and also tensor: