5 October 2004 !

Bénabou (1967) – invented "bicategories" (aka "weak 2-categories"), which have:

objects $\overset{x}{\bullet}$

morphisms $\overset{x}{\bullet} \longrightarrow \overset{y}{\bullet}$

2-morphisms

$$\begin{tikzcd} x \bullet \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\alpha \Downarrow"] & \bullet\, y \end{tikzcd}$$

– i.e. like categories but with 2-morphisms $\alpha : f \Rightarrow g$ between parallel 1-morphisms $f, g : x \to y$. As in category theory, we can compose 1-morphisms:

$$f : x \to y \;,\; g : y \to z \quad \text{give} \quad fg : x \to z$$

& we have identity 1-morphisms

$$1_x : x \to x$$

but these only satisfy associativity, left & right unit laws up to specified 2-isomorphism:

$$a_{f,g,h} : (fg)h \overset{\sim}{\Longrightarrow} f(gh)$$
$$\ell_f : 1_x f \overset{\sim}{\Longrightarrow} f$$
$$r_f : f 1_x \overset{\sim}{\Longrightarrow} f$$

What's a 2-isomorphism? Well, you can also compose 2-morphisms:

$$\alpha : f \Rightarrow g \quad \beta : g \Rightarrow h \quad \text{give} \quad \alpha\beta : g \Rightarrow h$$

& there are identity 2-morphisms:

$$1_f : f \Rightarrow f$$

which satisfy associativity, left & right unit laws "on the nose" — i.e. as equations. Thus we can talk about the inverse of a 2-morphism, and an invertible 2-morphism is called a <u>2-isomorphism</u>.

Also: a (associator), ℓ (left unit law), — should be called the "uniter"! r (right unit law) satisfy some coherence conditions:

$$\begin{tikzcd}
 & (f(gh))k \arrow[r, Rightarrow, "a_{f,gh,k}"] & f((gh)k) \arrow[dr, Rightarrow, "1_f \cdot a_{g,h,k}"] & \\
((fg)h)k \arrow[ur, Rightarrow, "a_{f,g,h} \cdot 1_k"] \arrow[dr, Rightarrow, "a_{fg,h,k}"'] & & & f(g(hk)) \\
 & (fg)(hk) \arrow[urr, Rightarrow, "a_{f,g,hk}"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
(f1_x)g \arrow[rr, Rightarrow, "a_{f,1_x,g}"] \arrow[dr, Rightarrow, "r_f \cdot 1_g"'] & & f(1_x g) \arrow[dl, Rightarrow, "1_f \cdot r_g"] \\
 & fg &
\end{tikzcd}$$

(and these two imply all other imaginable coherence laws)

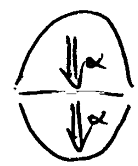
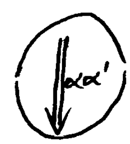
To make sense of these we also need an operation called "horizontal composition" of 2-morphisms:

Given

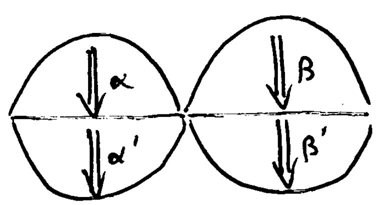
$$\begin{tikzcd} x \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\alpha\Downarrow"] & y \end{tikzcd} \qquad \begin{tikzcd} y \arrow[r, bend left=50, "f'"] \arrow[r, bend right=50, "g'"'] \arrow[r, phantom, "\alpha'\Downarrow"] & z \end{tikzcd}$$

we get

$$\begin{tikzcd} x \arrow[rr, bend left=40, "ff'"] \arrow[rr, bend right=40, "gg'"'] \arrow[rr, phantom, "\alpha\cdot\alpha'\Downarrow"] & & z \end{tikzcd}$$

(The previous form of composition —  gives 
— is called "vertical composition")

Lastly, we require the <u>interchange law</u> for horizontal & vertical composition:



$$(\alpha\cdot\beta)(\alpha'\cdot\beta') = (\alpha\alpha')\cdot(\beta\beta').$$

Bénabou had lots of examples in mind, e.g.:

- Cat is a bicategory with
  categories as objects
  functors as morphisms
  natural transformations as 2-morphisms.

(But this is a rather simplistic example — Cat is a
<u>strict</u> 2-category, meaning that the associator
and the unit laws are actually identity 2-morphisms)

- Mod is a bicategory with

  rings as objects
  (R, S) bimodules as morphisms from R to S
  (R, S) bimodule homomorphisms as 2-morphisms

  Composition of morphisms is <u>tensoring</u> of bimodules.
  We have an associator

  $$a_{M,N,O} : (M \otimes_S N) \otimes_T O \xrightarrow{\cong} M \otimes_S (N \otimes_T O)$$

  where $R \xrightarrow{M} S \xrightarrow{N} T \xrightarrow{O} U$.

- For any space X there's a bicategory $\Pi_2(X)$
  with
  points in X as objects
  paths in X as morphisms
  homotopy classes of homotopies
  between paths in X as 2-morphisms.

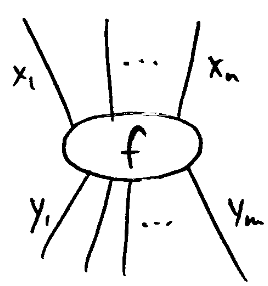
  (where we need "homotopy classes" at the top
  level to get the laws involving 2-morphisms
  to hold on the nose).

- Just as a monoid is a category with one object, a monoidal category is a bicategory with one object!

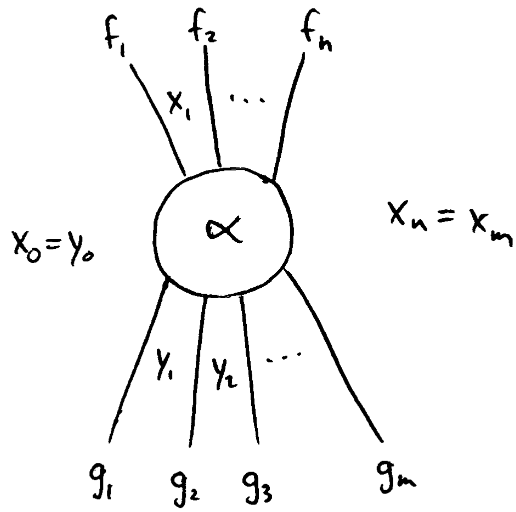
monoidal category	bicategory
morphisms	2-morphisms
objects	morphisms
———	one object

where "tensoring objects" in the monoidal category is secretly composing morphisms in the corresponding bicategory.

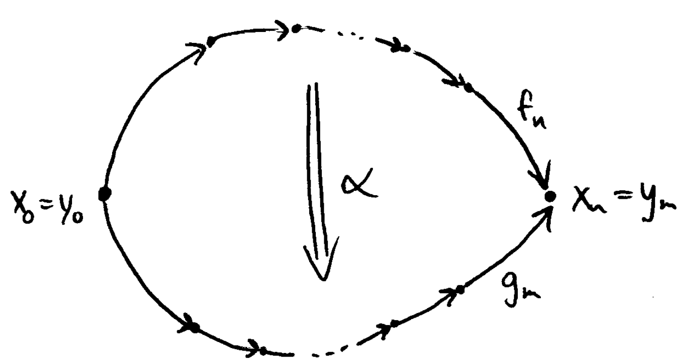
Recall that for monoidal categories we could draw a morphism $f : x_1 \otimes \cdots \otimes x_n \longrightarrow y_1 \otimes \cdots \otimes y_m$ as



More generally, we can draw 2-morphisms in a bicategory in the same fashion. Given composable morphisms $f_1 : x_0 \longrightarrow x_1$, $f_2 : x_1 \longrightarrow x_2, \ldots, f_n : x_{n-1} \longrightarrow x_n$ & $g_1 : y_0 \longrightarrow y_1, \ldots, g_m : y_{m-1} \longrightarrow y_m$, this is how we draw $\alpha: f_1 f_2 \cdots f_n \Longrightarrow g_1 g_2 \cdots g_m$ (if $x_0 = y_0$ & $x_n = y_m$)



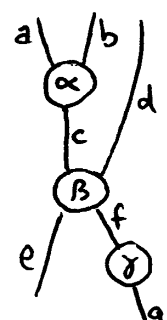
Note this is Poincaré dual to



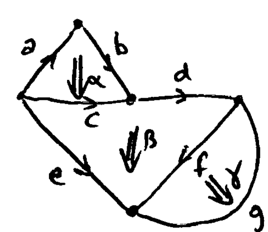
Given 2-morphisms

$\alpha : a\, b \Longrightarrow c$
$\beta : c\, d \Longrightarrow ef$
$\gamma : f \Longrightarrow g$

we can combine them using both vertical and horizontal composition



or



7 October 2004

Penrose (1971) — In GR physicists study "tensors", i.e. linear maps

$$f : \underbrace{V \otimes \cdots \otimes V}_{n} \longrightarrow \underbrace{V \otimes \cdots \otimes V}_{m}$$

& they use "index notation" to describe them. I.e. they ~~would~~ pick a basis e_i of V & say

$$f(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}) = \sum_{1 \le j_1, \ldots, j_m \le d} f^{j_1, \ldots, j_m}_{i_1, \ldots, i_n} \, e_{j_1} \otimes \cdots \otimes e_{j_m}$$

where $d = \dim V$.

Mathematicians would mock them (and still do) for indulging in this "debauch of indices," but these indices are needed to describe the most general ways of "composing" tensors. E.g. given

$$S : V \otimes V \longrightarrow V \otimes V \qquad S^{kl}_{ij}$$
$$T : V \longrightarrow V \otimes V \qquad T^{jk}_{i}$$

we can form all sorts of other tensors, e.g.: ordinary composition

$$ST : V \longrightarrow V \otimes V$$
$$(ST)^{jk}_{i} = S^{jk}_{lm} T^{lm}_{i}$$

(following the Einstein summation convention)

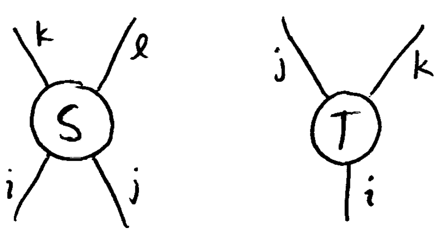
But we could also form

$$(T\otimes 1_V)S \;:\; V\otimes V \longrightarrow V\otimes V\otimes V$$

$$((T\otimes 1_V)S)_{ij}^{klm} = T_n^{kl}\, S_{ij}^{nm}$$

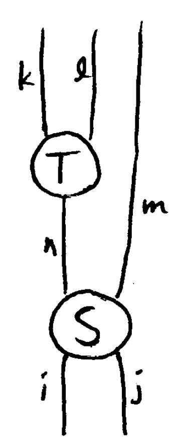
and many many more...

Penrose noticed that in these expressions, the indices don't <u>really</u> need to refer to a specific basis of a vector space. — the results are basis-independent; the indices only say how the tensors are hooked up. This is his <u>"abstract index notation."</u> He also started drawing tensors as "black boxes"



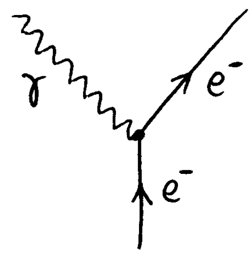
& composing them diagrammatically:

$S\circ T =$ 

$(T\otimes 1_V)\circ S =$ 

And we can now even leave out the indices — the 'wires' do the work.

Nowadays we know this notation works in any monoidal category, or even a bicategory, where they're called "string diagrams". Penrose didn't know about monoidal categories, but he <u>did</u> know about Feynman diagrams, and he realized what the relationship was.



In Feynman diagrams, each kind of particle has a specific vector space (Hilbert space) of states; these are objects in Vect. The vertices of Feynman diagrams describe interactions; these are morphisms in Vect. In the theory of Feynman diagrams, all the vector spaces are actually representations of some symmetry group G (e.g. Poincaré group). Also, the vertices actually represent morphisms between representations — i.e. <u>intertwiners</u>. In short, our monoidal category is not Vect but $\mathrm{Rep}(G)$ — the category of representations of G.

Penrose played the same game but taking $G = SU(2)$. This group has one irrep of each dimension, called (by physicists) the "spin-0" (= 1d), "spin-$\frac{1}{2}$" (= 2d), "spin-1" (= 3d), ... representations.