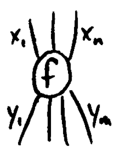
12 October 2004

A summary so far:

Monoidal Category		G = trivial gp (Rep(G) = Vect)	$G \geq$ Poincaré	$G = GL(n)$	$G = G$	SU(2)
objects x	$\vert\, x$	vector spaces	particles	tensors	representations	"spins"
morphisms $f: x_1 \otimes \cdots \otimes x_n \to y_1 \otimes \cdots \otimes x_m$		linear transformations	interactions	intertwiners	intertwiners	?

Rep(G) as a monoidal category

Penrose considered the case $G = SU(2)$. In the Fall 2000 QG Seminar notes we saw that:

- there's one irrep of $SU(2)$ of each dimension $1, 2, 3, \ldots$ — said to be of "spin" $0, \frac{1}{2}, 1, \ldots$
- there's a one-dimensional space of intertwiners

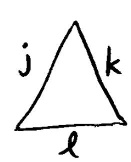
$$f : [j] \otimes [k] \longrightarrow [\ell]$$

([j] being the spin-j rep) when

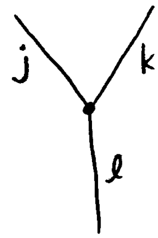
$$|j-k| \leq \ell \leq j+k \quad \& \quad j+k+\ell \in \mathbb{Z};$$

there's a zero-dimensional space of intertwiners otherwise.

This inequality is the triangle inequality:
It says you can draw a triangle like this $\longrightarrow$

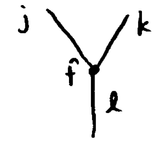


The Poincaré dual of this triangle is



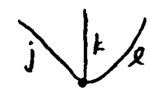
Given an intertwiner

$$f : [j] \otimes [k] \longrightarrow [\ell]$$



we can get

$$\tilde{f} : [j] \otimes [k] \otimes [\ell]^* \longrightarrow \mathbb{C}$$



in the canonical way, but $[\ell]^* \cong [\ell]$ in a canonical way for any rep. of $SU(2)$.

We can also get

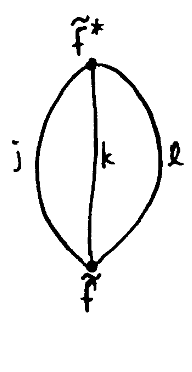
$$\tilde{f}^* : \mathbb{C}^* \longrightarrow [j]^* \otimes [k]^* \otimes [\ell]^*$$

or

$$\tilde{f}^* : \mathbb{C} \longrightarrow [j] \otimes [k] \otimes [\ell]$$

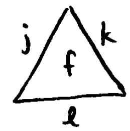
since reps of $SU(2)$ are iso. to their duals (and so is $\mathbb{C}$ — $(c \in \mathbb{C}) \longmapsto (z \mapsto cz)$)

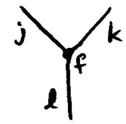
We pick our favorite $f : [j] \otimes [k] \to [\ell]$ by demanding



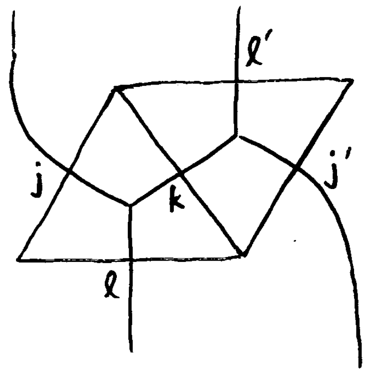
$\tilde{f}^* \tilde{f} : \mathbb{C} \to \mathbb{C}$ is just multiplication by 1.

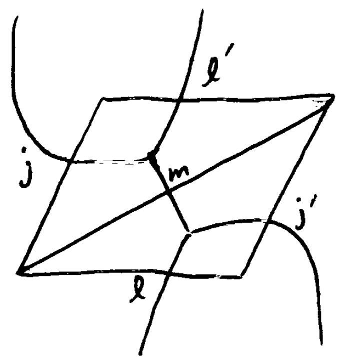
(This determines f up to a sign, which ambiguity we can either eliminate by more work or tolerate)

Starting from f = 

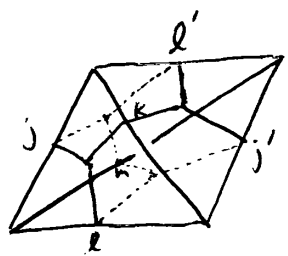
= 

we can build other morphisms



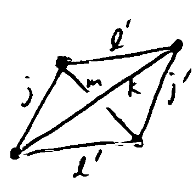


we can assemble these two quadrilaterals into a tetrahedron



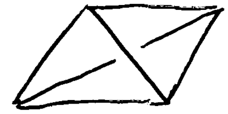
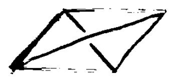
(note; the dual of a tetrahedron is a tetrahedron)

or:



This is an intertwiner from $\mathbb{C}$ to $\mathbb{C}$, i.e. (multiplication by) a complex number

T. Regge & G. Ponzano (1968) — applied Penrose's spin networks (before he invented them!) to create a theory of quantum gravity in which spacetime was 2+1 dimensional built from little tetrahedra.

In this theory, the amplitude for this tetrahedron  is given by the number 

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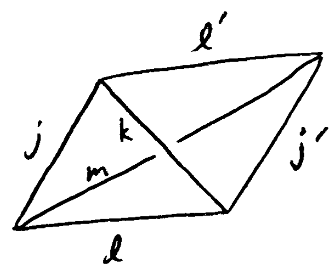
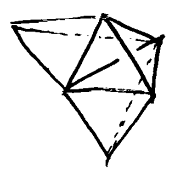
Ponzano & Regge called this number:

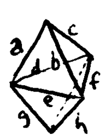


$j, j', k, l, l', m = 0, \frac{1}{2}, 1, \ldots$

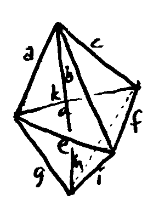
the "6j symbols" – it had been studied in quantum chemistry before. They related its asymptotics for large spins to the action in 3d general relativity. Their idea was: given a 3d spacetime chopped into tetrahedra:



we can describe its geometry by specifying the edge lengths, which take discrete values $0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. Then you calculate the relative amplitude for spacetime to have this shape by taking the product of 6j symbols over all tetrahedra, times some extra fudge factor for each triangle, to achieve this:



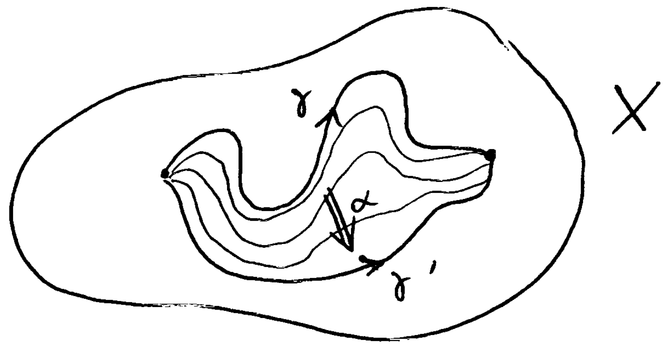
$$= \sum_{k=0,\frac{1}{2},\ldots}$$



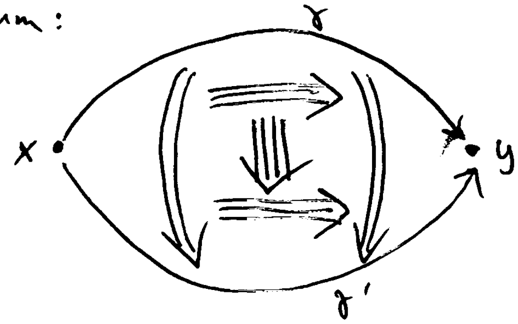
The <u>Eliot-Biedenharn identity</u>

only finitely many nonzero terms in this sum because of the triangle inequality.

Grothendieck (1983) – In his 600-page letter to Dan Quillen, he pondered the idea that to any space X you could associate an ω-category, its "fundamental ω-groupoid" $\Pi_\omega(X)$



which has points of X as its objects, paths in X as morphisms, homotopies of paths as 2-morphisms, and so on ad infinitum:



All the ways of composing these j-morphisms should give $\Pi_\omega(X)$ the algebraic structure of an "ω-category." All laws governing composition should hold only up to higher morphisms, which satisfy laws of their own up to still higher morphisms, etc.

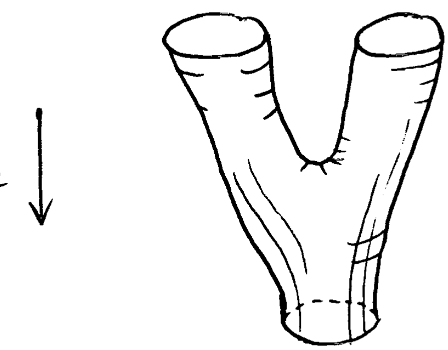
$\Pi_{\varpi}(X)$ should be a complete invariant of homotopy types.
That is:

$$X \text{ homotopy equiv. to } Y \iff \Pi_{\varpi}(X) \text{ is } \varpi\text{-equivalent to } \Pi_{\varpi}(Y)$$

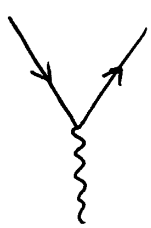
This dream is currently under construction...

String Theory (1980's –) generalized the concept of Feynman diagrams to include 2-dimensional diagrams like

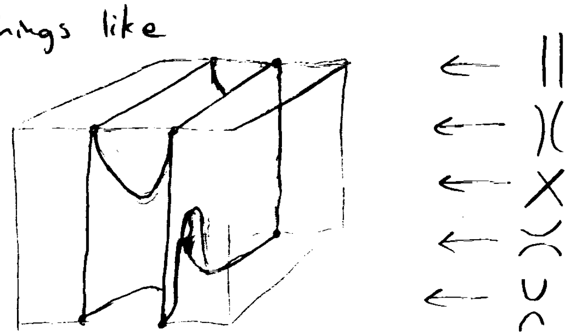
time



can mimic:

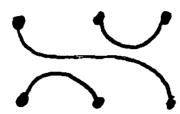


It turns out that "open" strings are also interesting, so we can get things like



i.e. a 2-category with

- bunches of points as objects • • •
- bunches of strands as morphisms:



- bunches of surfaces as 2-morphisms

In fact this is something like a monoidal category, or more...

Also, just as we get a term in the action ~~by~~ of a charged particle by integrating the 1-form A (electromagnetic vector potential) over its worldline:

γ $\qquad\qquad \int_\gamma A$

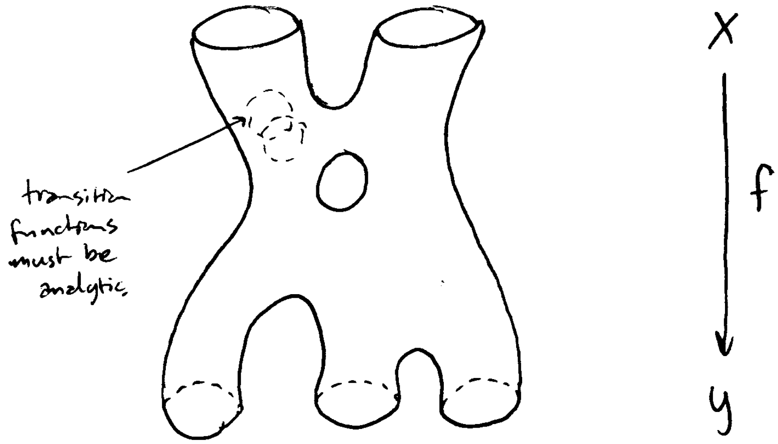
we can integrate a 2-form B (the Kalb-Ramond field) over the worldsheet of a string to get a term in its action.

Graeme Segal (1989) — formalized the concept of "a string theory," by inventing conformal field theories (CFT).

A CFT is, roughly, a functor

$$Z : 2\mathrm{Cob}_{\mathbb{C}} \longrightarrow \mathrm{Hilb}$$

where $2\mathrm{Cob}_{\mathbb{C}}$ is a category whose objects are bunches of circles & morphisms are 2-dimensional cobordisms:



equipped with the structure of complex manifolds!

Z assigns a Hilbert space of states $Z(x)$ to any bunch of circles x & a "time evolution" operator

$$Z(f): Z(x) \longrightarrow Z(y)$$

to any "string worldsheet" f.

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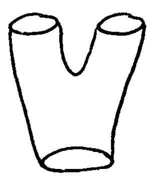
Actually, a CFT is not quite just any functor

$$Z : 2\mathrm{Cob}_{\mathbb{C}} \longrightarrow \mathrm{Hilb}$$



$$\varphi \quad \begin{array}{c} x \\ \downarrow \\ y \end{array} \quad \Bigg| \quad \begin{array}{c} Z(x) \\ \downarrow \\ Z(y) \end{array}$$

It's more like a <u>symmetric monoidal</u> functor. Roughly:

$2\mathrm{Cob}_{\mathbb{C}}$ is a monoidal category with disjoint union as tensor product $\otimes$;
Hilb is a monoidal category with $\otimes$ the usual tensor product of Hilb. spaces
& Z preserves the tensor product up to specified isomorphism.

E.g. $Z(\bigcirc\ \bigcirc) \cong Z(\bigcirc) \otimes Z(\bigcirc)$

via a specific isomorphism. (state of a pair of strings is isomorphic to the product of states of the strings)

Furthermore, $2\mathrm{Cob}_{\mathbb{C}}$ is a <u>symmetric</u> monoidal category with

$$S_{a,b} : a \otimes b \longrightarrow b \otimes a$$

given by