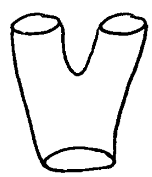
19 October 2004

Actually, a CFT is not quite just any functor

$$Z : 2\mathrm{Cob}_{\mathbb{C}} \longrightarrow \mathrm{Hilb}$$



$$\varphi \quad \begin{array}{c} x \\ \big\downarrow \\ y \end{array} \quad \Bigg| \quad \begin{array}{c} Z(x) \\ \big\downarrow \\ Z(y) \end{array}$$

It's more like a <u>symmetric monoidal</u> functor. Roughly:

$2\mathrm{Cob}_{\mathbb{C}}$ is a monoidal category with disjoint union as tensor product $\otimes$;
Hilb is a monoidal category with $\otimes$ the usual tensor product of Hilb. spaces
& Z preserves the tensor product up to specified isomorphism.

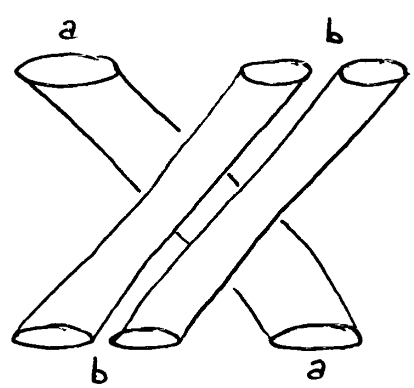
E.g. $Z(\bigcirc\ \bigcirc) \cong Z(\bigcirc) \otimes Z(\bigcirc)$

via a specific isomorphism. (state of a pair of strings is isomorphic to the product of states of the strings)

Furthermore, $2\mathrm{Cob}_{\mathbb{C}}$ is a <u>symmetric</u> monoidal category with

$$S_{a,b} : a \otimes b \longrightarrow b \otimes a$$

given by



and Hilb is a symmetric monoidal category with

$$S_{H,H'} : H \otimes H' \xrightarrow{\cong} H' \otimes H$$
$$\psi \otimes \varphi \longmapsto \varphi \otimes \psi$$

and Z preserves the symmetry:

$$\begin{tikzcd}
Z(a\otimes b) \arrow[d, "Z(S_{a,b})"', "\cong"] & & Z(a)\otimes Z(b) \arrow[ll, "\cong"'] \arrow[d, "S_{Z(a),Z(b)}"', "\cong"] \\
Z(b\otimes a) & & Z(b)\otimes Z(a) \arrow[ll, "\cong"']
\end{tikzcd}$$

where the horizontal arrows come from the definition of monoidal functor.

(Actually there are further subtleties: Identity morphisms in $2\mathrm{Cob}_{\mathbb{C}}$ must be "zero length" cylinders, & we're only defining CFT's with "central charge 0" – e.g. all good string theories. (the reason string theory only works in 10 dimensions is that in other dimensions one doesn't get a CFT with central charge 0))

Atiyah (1989) – invented the definition of a <u>topological quantum field theory</u>, or TQFT by modifying Segal's definition:

A TQFT is a symmetric monoidal functor

$$Z: n\mathrm{Cob} \longrightarrow \mathrm{Hilb}$$

where nCob is the symmetric monoidal category with (smooth compact oriented) $(n-1)$-manifolds as objects, & cobordisms between these as morphisms. Later we'll construct lots of 2d and 3d TQFTs.

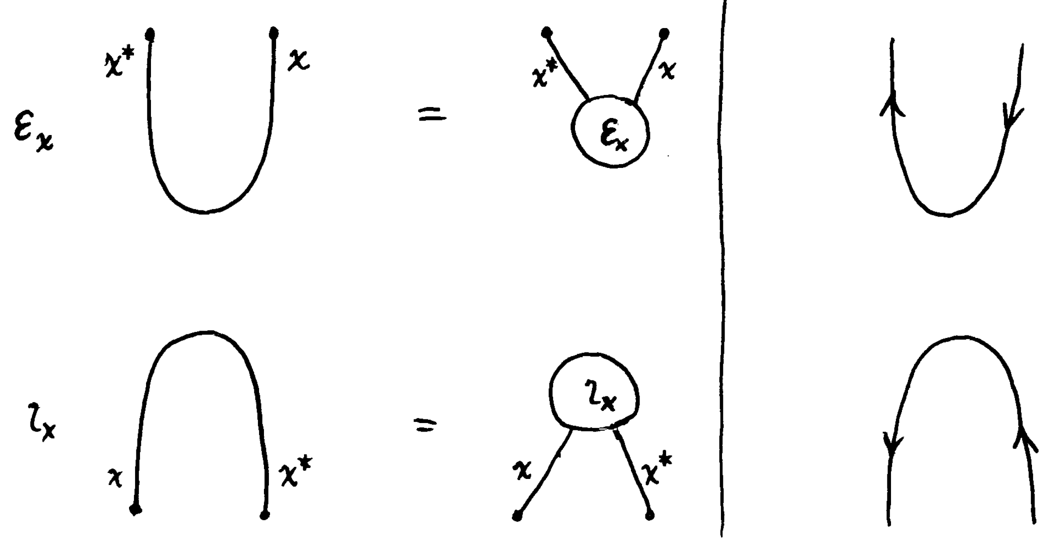
Both nCob and Hilb have duals of objects (if Hilb means finite-dimensional Hilbert spaces). An object x in a monoidal category has a dual x^* if there are morphisms:

$$\varepsilon_x : x^* \otimes x \longrightarrow 1 \qquad \text{(counit)} \qquad \left(\varepsilon \text{ for "evaluation"}\right)$$

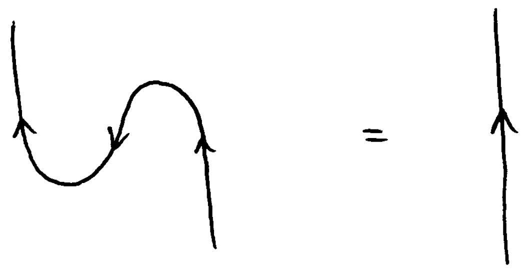
and

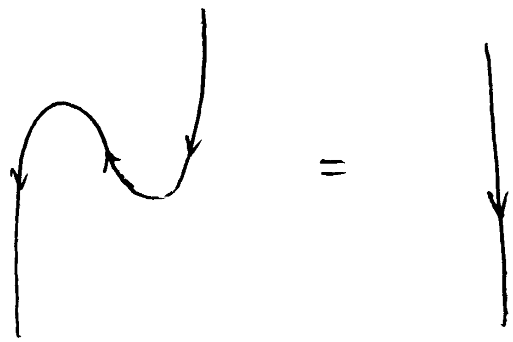
$$\imath_x : 1 \longrightarrow x \otimes x^* \qquad \text{(unit)} \qquad \left(\imath \text{ for "identity matrix"}\right)$$

which we can draw as | or just as:



(like annihilation and creation of particle pairs) satisfying:





Indeed, in Hilb, H^* is just the dual vector space of H made into a Hilbert space,

$$\varepsilon_H : H^* \otimes H \longrightarrow \mathbb{C}$$
$$f \otimes \psi \longmapsto f(\psi)$$

$$\iota_H : \mathbb{C} \longrightarrow H \otimes H^* \cong \mathrm{End}(H)$$
$$\alpha \longmapsto \alpha 1_H$$

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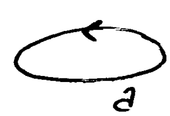
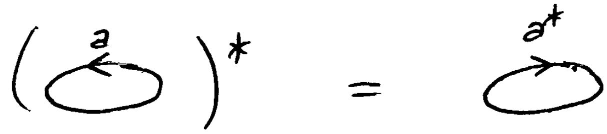
Similarly, nCob has duals for objects. An object $a \in$ nCob is a compact oriented $(n-1)$-manifold:

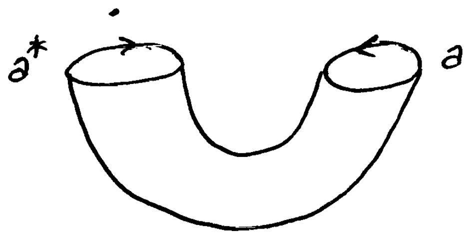


Its dual is just the same $(n-1)$-manifold with reversed orientation :



since we have

$$\varepsilon_a : a^* \otimes a \longrightarrow 1$$



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and also

$$\iota_a : 1 \longrightarrow a \otimes a^*$$

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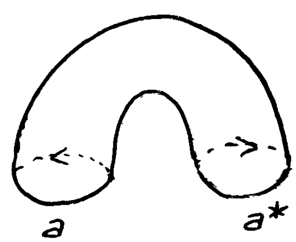
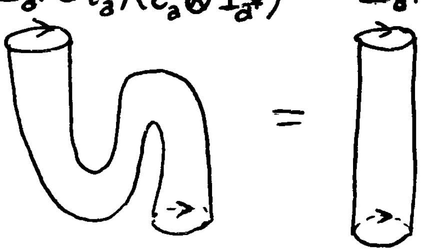
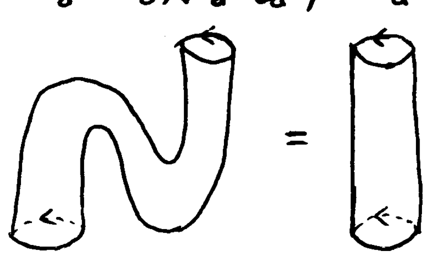


satisfying the desired laws:

$$(1_{a^*} \otimes \iota_a)(\varepsilon_a \otimes 1_{a^*}) = 1_{a^*} \qquad (\iota_a \otimes 1_a)(1_a \otimes \varepsilon_{a^*}) = 1_a$$



and



zigzag identity

zagzig identity

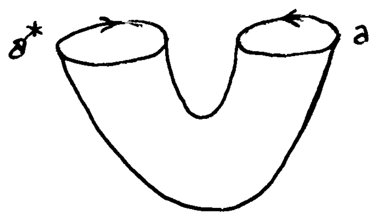
A monoidal functor automatically preserves duals of objects (up to specified isomorphism) so a TQFT

$$Z : n\mathrm{Cob} \to \mathrm{Hilb}$$

will satisfy

$$Z(a^*) \cong Z(a)^*$$

via a specific isomorphism.



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$$\overset{Z}{\longmapsto}$$

$$\begin{tikzcd} Z(a^* \otimes a) \arrow[d] \arrow[r, phantom, "\cong"] & Z(a)^* \otimes Z(a) \arrow[d, "\varepsilon_{Z(a)}"] \\ Z(1) \arrow[r, phantom, "\cong"] & \mathbb{C} \end{tikzcd}$$

(dual pairing)

note: states are complex numbers/ complex numbers. — Only one state for the empty universe.

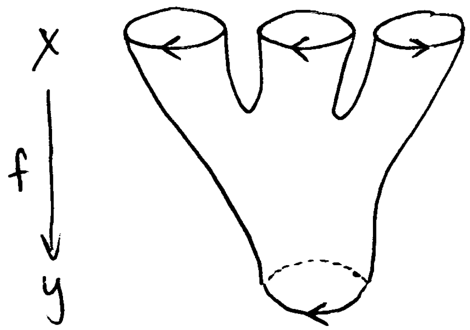
Besides duals of objects (which for $n\mathrm{Cob}$ comes from reversing the orientation of space) both $n\mathrm{Cob}$ & Hilb have duals of morphisms (which for $n\mathrm{Cob}$ comes from reversing the orientation of time).

A category C has duals for morphisms if it's equipped with a contravariant endofunctor $* : C \to C$ that's the identity on objects & has $*^2 = 1_C$. I.e., $f : x \to y$ in C gives $f^* : y \to x$ in C such that $(fg)^* = g^* f^*$ and $1_x^* = 1_x$ $\forall x \in C$ (actually can derive this from ↑) and $f^{**} = f$.

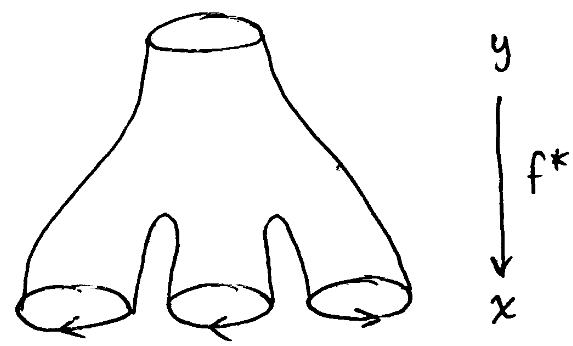
In Hilb, the dual of a linear operator $f: H \to H'$ is its Hilbert space adjoint $f^*: H' \to H$ defined by

$$\langle \psi, f\varphi \rangle = \langle f^*\psi, \varphi \rangle \quad \forall \varphi \in H \;\; \psi \in H'$$

In nCob the dual of a cobordism $f: x \to y$



is



defined by reversing the time direction.

It's not true that a TQFT

$$Z: n\mathrm{Cob} \longrightarrow \mathrm{Hilb}$$

preserves duals for morphisms, but the ones that do are physically interesting & are called unitary TQFTs:

$$Z(f^*) = Z(f)^*.$$

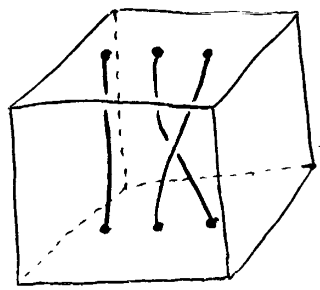
André Joyal & Ross Street (1986) — invented the concept of a braided monoidal category. These are monoidal categories equipped with a natural isomorphism

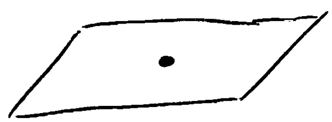
$$B_{x,y}: x \otimes y \xrightarrow{\sim} y \otimes x$$

satisfying all laws of a symmetric monoidal category <u>except</u>

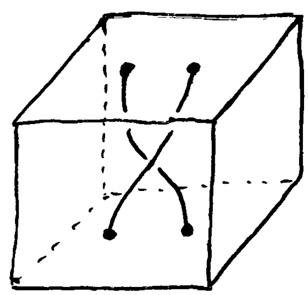
$$B_{x,y} B_{y,x} = 1_{x \otimes y}$$

The name comes from the example Braid, where morphisms look like:

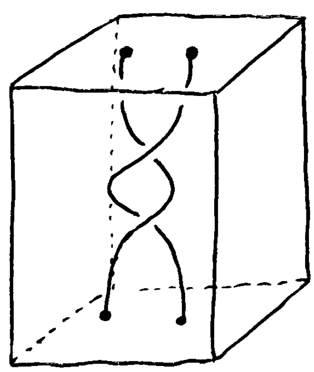
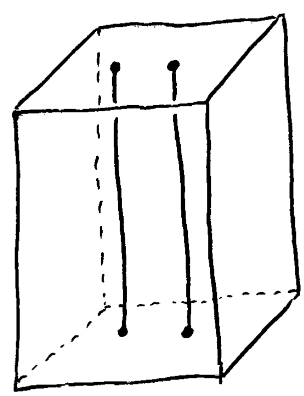


Here if $x =$ 

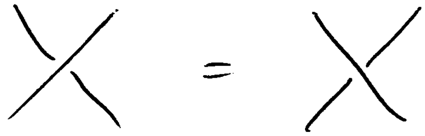
then

$B_{x,x} =$ 

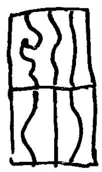
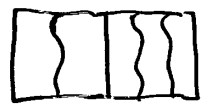
&

$B_{x,x} B_{x,x} =$  $\neq$  $= 1_{x \otimes x}$

Braids in 4 dimensions form a symmetric monoidal category:



Braids in 2 dimensions form a monoidal category:

composition  &  tensor prod.

Braids in 1 dimension form a category:

So we see a pattern:

1d	category
2d	monoidal category
3d	braided monoidal category
4d	symmetric monoidal category
5d	symmetric monoidal category
⋮	⋮
nd	symmetric monoidal category
⋮	⋮