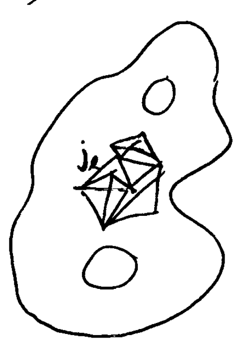
26 October 2004

Witten (1989) — described how the symmetric monoidal category Rep(SU(2)) can be "deformed" (as a function of $\hbar \in \mathbb{R}$ or $q = e^{i\hbar} \in U(1)$) into a braided but nonsymmetric monoidal category "Rep($SU_q(2)$)" where $SU_q(2)$ is a <u>quantum group</u> (<u>not</u> a group, but a thing whose representations form a <u>braided</u> monoidal category), already discovered by Fadeev, Drinfeld, & the Russian school (working on completely integrable systems). The new thing about Witten's approach is that it got ahold of the braided monoidal category <u>directly</u> (i.e. without knowing about $SU_q(2)$), using CFT.

Turaev & Viro (1992) — Showed how the Ponzano-Regge model, defined using Rep(SU(2)) could be improved by using Rep($SU_q(2)$) instead! (Actually, they only learned of the Ponzano-Regge model and its relation to 3d quantum gravity <u>later</u>.)



triangulated manifold
with edges labelled by spins

In the P-R model we can calculate an amplitude for any triangulated compact 3-manifold with edge lengths given by spins $j = 0, \frac{1}{2}, \ldots$. Alas, the sum of these

~~of these~~
amplitudes over all labellings diverges. But $SU_q(2)$ has only finitely many irreps: $j = 0, \frac{1}{2}, \ldots, \frac{k}{2}$ where

$$q = e^{\frac{2\pi i}{k+2}}.$$

So: in Turaev & Viro's version of the P-R model, the sum over labellings is a finite sum! Even better, the result doesn't depend on the choice of triangulation! Even better, we get a TQFT

$$Z : 3\mathrm{Cob} \longrightarrow \mathrm{Hilb}$$

which gives these amplitudes for compact 3-manifolds.

Barrett & Westbury (1992) — showed that Turaev & Viro's recipe for getting a 3d TQFT from $\mathrm{Rep}(SU_q(2))$ generalized to a large class of monoidal categories — called "spherical categories." (In fact they did this before reading Turaev & Viro's work, & unlike T-V, they knew the relation to 3d QG.)

Fukuma, Hosono, Kawai (1992) — used a similar trick to get 2d TQFTs:

$$Z : 2\mathrm{Cob} \longrightarrow \mathrm{Hilb}$$

from certain nice monoids (in fact semisimple algebras)

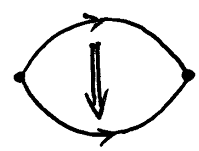
Louis Crane (1993) — Published a paper entitled "Categorical Physics" in which he explicated this pattern:

monoids	2d TQFTs
monoidal categories	3d TQFTs
monoidal 2-categories?	4d TQFTs?
⋮	⋮

in which <u>increase in dimension</u> goes along with <u>categorification</u>

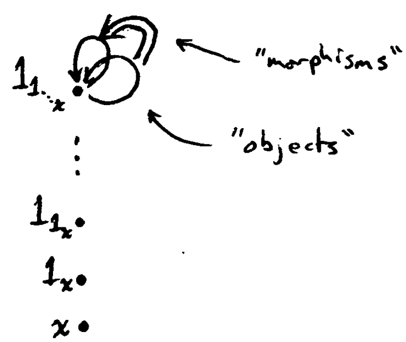
Cantor
1870s

Eilenberg-MacLane
1940s



Benabou
1960s

Baez & Dolan (1995) — considered $(n+k)$-categories with only one object, one morphism, ..., one $(k-1)$-morphism



& arbitrary from then on. Such a thing is an n-category with extra structure. E.g. a 1-category with just one object $(n=0, k=1)$ is a special sort of 0-category (set), namely a monoid. In general we call these "k-tuply monoidal n-categories":

	$n=0$	$n=1$	$n=2$
$k=0$	sets	categories	2-categories
$k=1$	monoids	monoidal categories	monoidal 2-categories
$k=2$	commutative monoids	braided monoidal categories	braided monoidal 2-categories
	commutative monoids	symmetric monoidal categories	sylleptic monoidal 2-category
			symmetric monoidal 2-categories

syllepsis