2 November 2004

2D TQFTs from SEMISIMPLE ALGEBRAS

I'll now describe how Fukuma, Hosono & Kawai (hep-th/9212154) constructed TQFTs

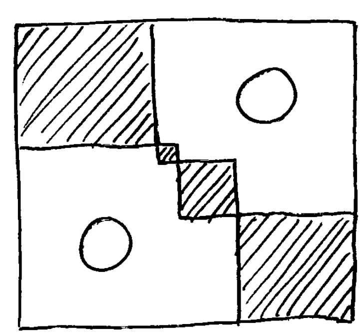
$$Z : 2\mathrm{Cob} \longrightarrow \mathrm{Vect} \qquad \text{(symm. mon. functor)}$$

from (finite-dimensional) semisimple (associative) algebras (over $\mathbb{C}$) — i.e. algebras which are direct sums of <u>simple</u> algebras, namely those without nontrivial ideals. (Recall, an <u>ideal</u> I of an algebra A is a vector subspace $I \subseteq A$ s.t. $AI \subseteq I$ & $IA \subseteq I$)

<u>Wedderburn's Theorem</u> says any (fin. dim) simple algebra over $\mathbb{C}$ is isomorphic to $M_n(\mathbb{C})$ – the algebra of $n \times n$ complex matrices. So a typical semisimple algebra is:

$$\underset{\mathbb{C}}{\underset{\shortparallel}{M_1(\mathbb{C})}} \oplus \underset{\mathbb{C}}{\underset{\shortparallel}{M_1(\mathbb{C})}} \oplus M_{497}(\mathbb{C}) \oplus M_{1000}(\mathbb{C})$$

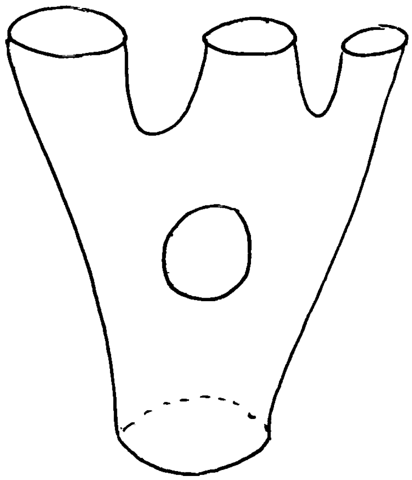
So they're all just algebras of block diagonal matrices:



Here's the idea:

morphism in 2Cob

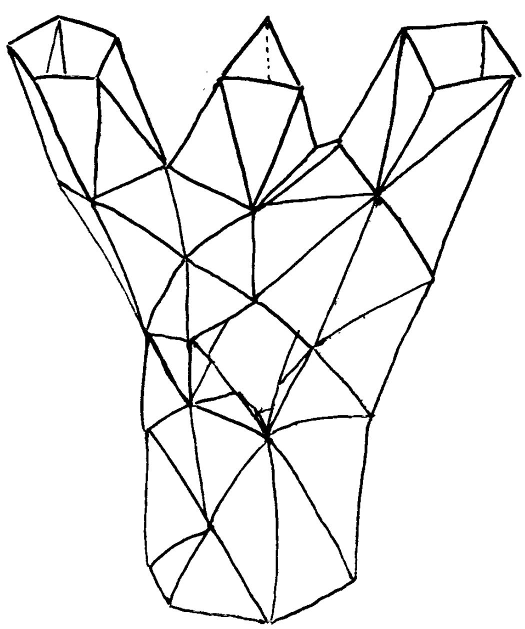
$$\begin{array}{c} S \\ \Big\downarrow M \\ S' \end{array}$$



should give

$$\begin{array}{c} Z(S) \\ \Big\downarrow Z(M) \\ Z(S') \end{array}$$

To get this, the first step is to triangulate the (compact, oriented) 1-manifolds S & S' & the (compact, oriented) cobordism M:



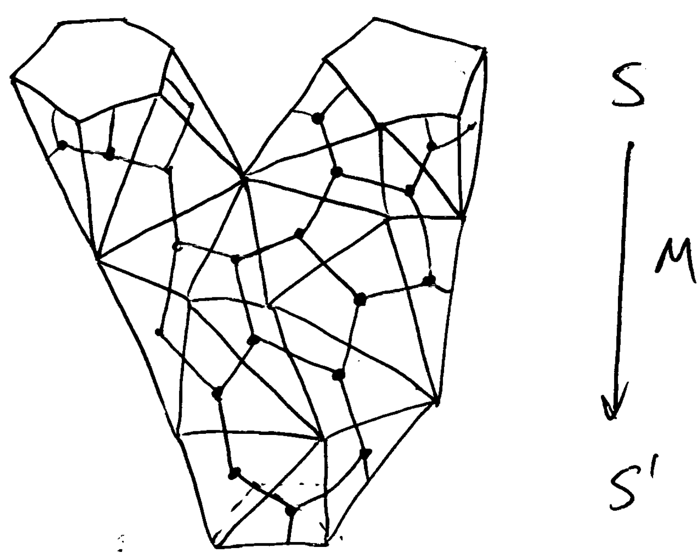
Then we'll get vector spaces $\tilde{Z}(S)$ & $\tilde{Z}(S')$ & a linear operator

$$\tilde{Z}(M) : \tilde{Z}(S) \longrightarrow \tilde{Z}(S')$$

depending on the choice of triangulation, & use this to define a triangulation-independent

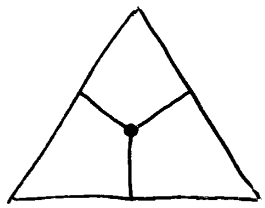
$$Z(M) : Z(S) \longrightarrow Z(S').$$

To get $\tilde{Z}$, we'll draw a trivalent graph (each vertex has 3 edges) Poincaré dual to our triangulated M and interpret this as a Feynman diagram



& get the linear operator $\tilde{Z}(M)$ out of this.

To interpret this graph as a Feynman diagram we need to label each edge with a vector space & each vertex with a linear operator. We'll use the same vector space A for every edge and the same operator $m: A \otimes A \longrightarrow A$ for every vertex

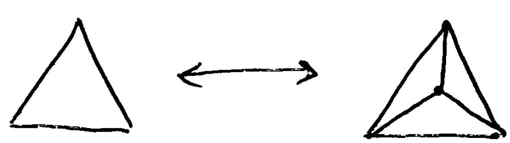


$$\begin{array}{c} A \otimes A \\ \downarrow m \\ A \end{array}$$

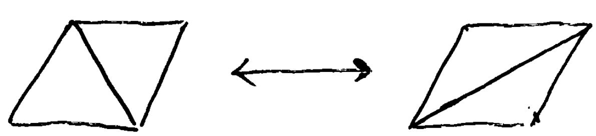
(at least for triangles with two "input" edges & one "output" edge, whatever that means).

Ultimately we want $Z(M)$, a triangulation-independent operator. For this, it's helpful to know <u>Alexander's Thm.</u>, which gives a sufficient set of "moves" to go between any two triangulations of a compact **2-manifold**:

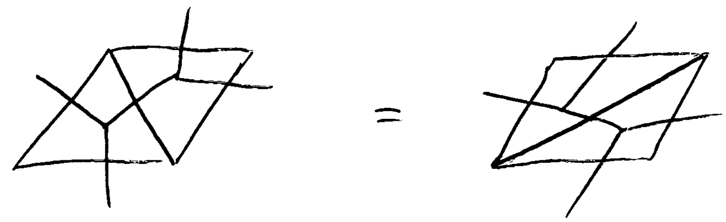
The 1-3 move and its inverse



& The 2-2 move



The 2-2 move, applied to $m: A \otimes A \to A$ says



$$\begin{array}{ccc} A \otimes A \otimes A & & A \otimes A \otimes A \\ \downarrow{\scriptstyle 1 \otimes m} & & \downarrow{\scriptstyle m \otimes 1} \\ A \otimes A & & A \otimes A \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} \\ A & & A \end{array}$$

m must be <u>associative</u> for these 2 operators to be the same!

How about the 1-3 move? (It's semisimplicity & mult. unit)

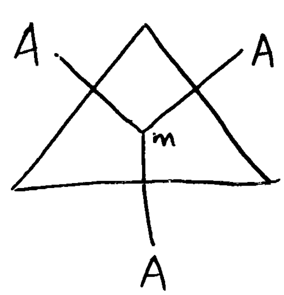
<u>4 November 2004</u>

Recall...

Idea: we want a linear operator

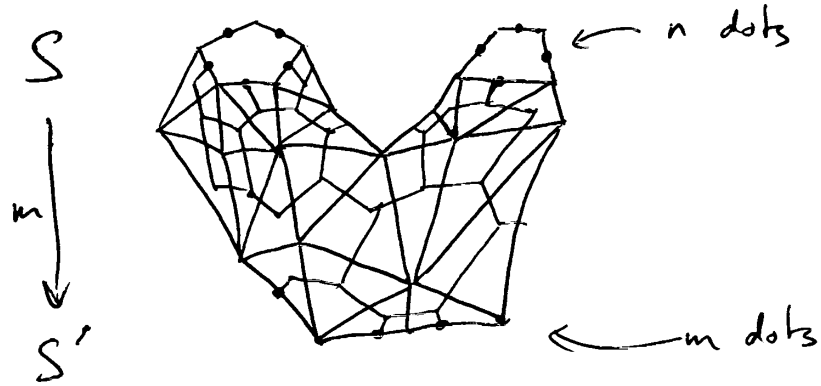
$$\tilde{Z}(M): \tilde{Z}(S) \longrightarrow \tilde{Z}(S')$$

from any triangulated 2d cobordism, obtained by choosing a vector space A & a linear operator $m: A \otimes A \to A$:

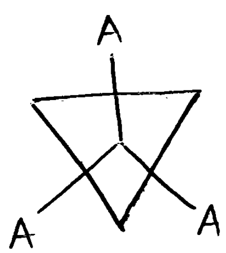


and then reading the graph dual to the triangulation of M as a linear operator

$$\begin{array}{ccccc} \tilde{Z}(M): & \tilde{Z}(S) & \longrightarrow & \tilde{Z}(S') \\ & \| & & \| \\ & A^{\otimes n} & & A^{\otimes m} \end{array}$$



Problem: what operator do we use for



$$\begin{array}{c} A \\ \downarrow \\ A \otimes A \end{array}$$

The best idea is to choose an isomorphism $A \cong A^*$ and let this operator be:

$$\begin{array}{c} A \\ \downarrow \wr \\ A^* \\ m^\dagger \downarrow \\ A^* \otimes A^* \\ \downarrow \wr \\ A \otimes A \end{array}$$

where $f^\dagger : V^* \longrightarrow W^*$ is the adjoint of the linear operator $f: W \to V$

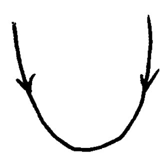
But how do we choose the isomorphism $A \cong A^*$?
One way is to choose a bilinear map

$$g : A \otimes A \longrightarrow \mathbb{C}$$

which is nondegenerate: $g(v,w) = 0 \ \forall w \Rightarrow v = 0$,
which is equivalent (when A is finite dimensional) to

$$\begin{array}{ccc} A & \longrightarrow & A^* \\ v & \longmapsto & g(v,-) \end{array}$$

being bijective (it's always injective, and $\dim A = \dim A^*$), i.e. an isomorphism. We can draw g as:

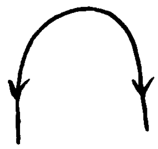


g

$$\begin{array}{c} A \otimes A \\ \downarrow \\ \mathbb{C} \end{array}$$

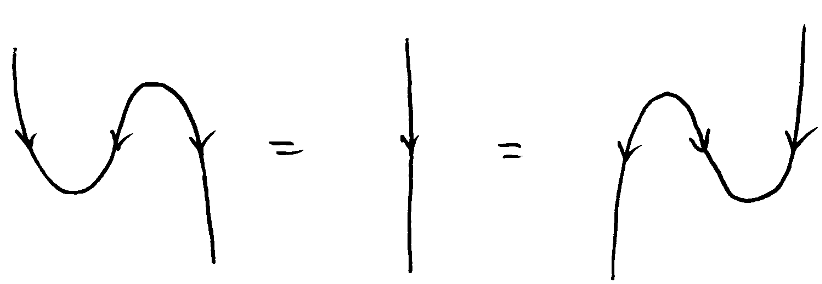
and it turns out that g is nondegenerate iff there's a map

$\bar{g}$

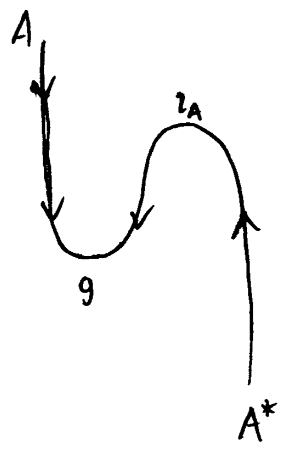


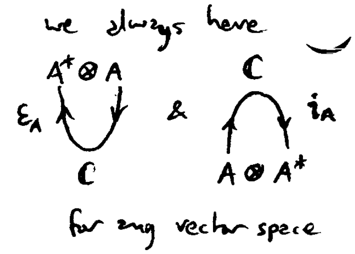
$$\begin{array}{c} \mathbb{C} \\ \downarrow \\ A \otimes A \end{array}$$

s.t.

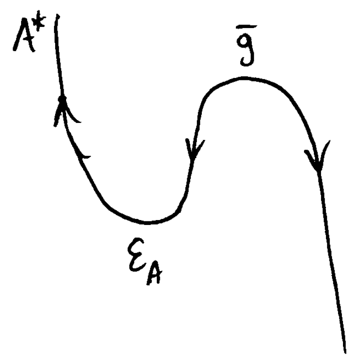


Indeed this gives us an isomorphism $\#: A \to A^*$:



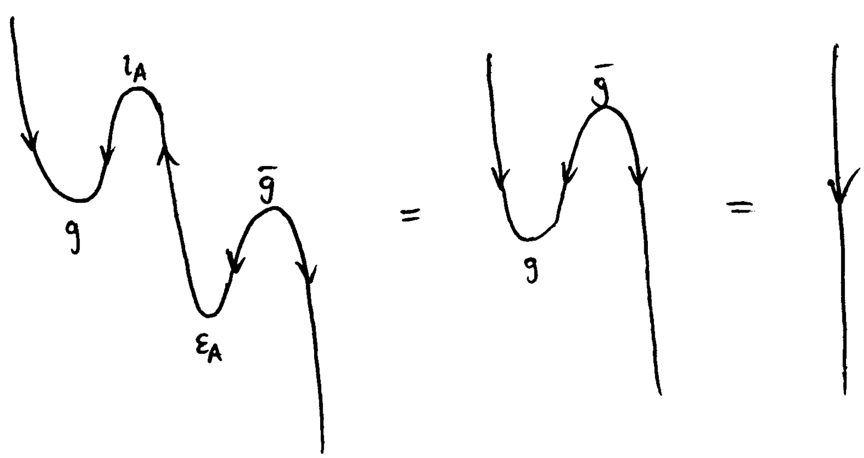


This is an isomorphism because it has inverse



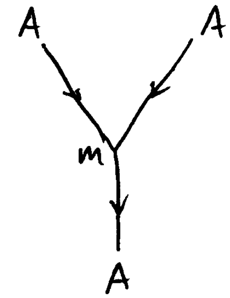
called $\flat: A^* \longrightarrow A$. It's easy to show $\#\flat = \flat\# = 1$ (=4?)

e.g.

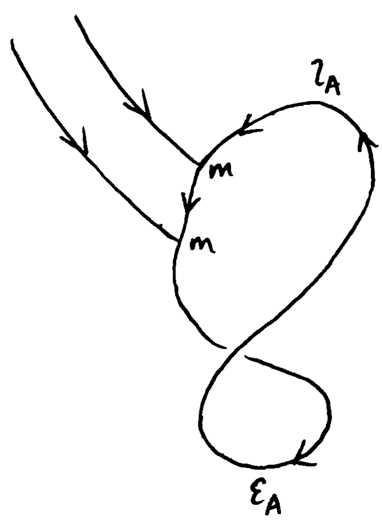


(and the other way)

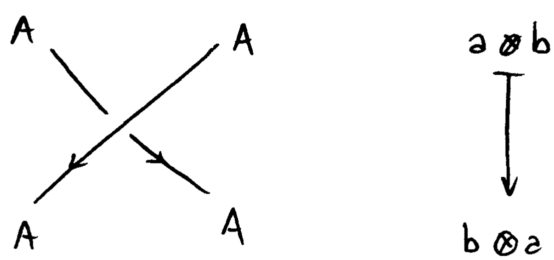
But how do we choose a nondegenerate bilinear form on A?
There's a God-given bilinear form on any algebra A!
In an algebra we have



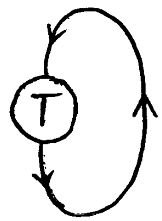
and this alone lets us define



using the fact that Vect is a symmetric monoidal category, which gives us:

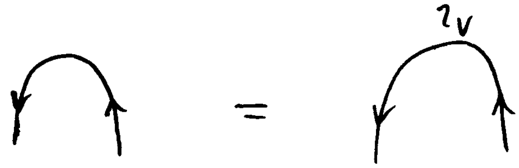


In fact this g will be nondegenerate iff A is semisimple! But what $\underline{\text{is}}$ this g, really? Given a linear operator $T : V \longrightarrow V$ we can define $tr(T) = \sum_i T^i_i$ or diagrammatically:

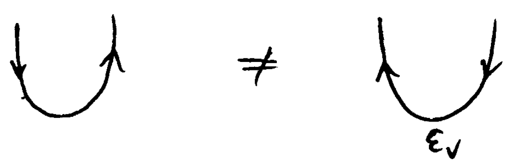


note: sum over internal edges just like Feynman diagrams

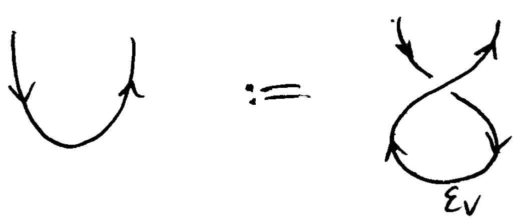
Note here



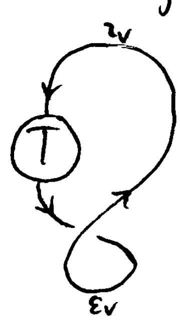
but



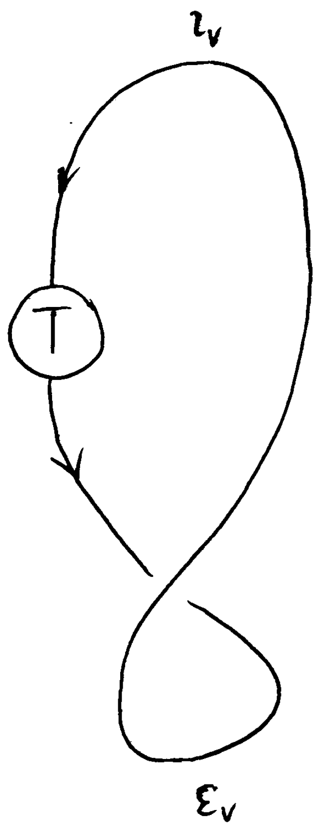
but it's okay if we define



In other words given $T : V \longrightarrow V$ (any morphism in a symmetric monoidal category with duals for objects) we can define:

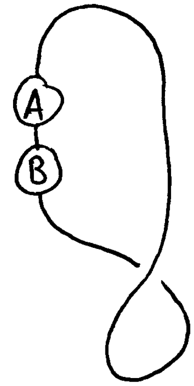
$tr(T) =$ 

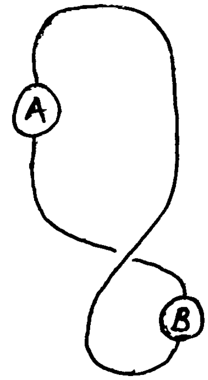
and $tr(T) : 1 \to 1$ (i.e. a <u>number</u> in our example of Vect)

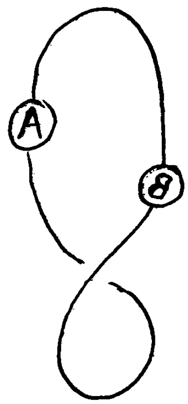


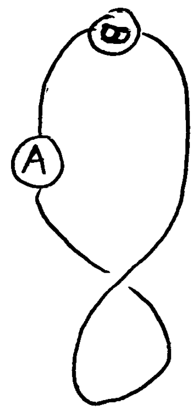
$$
\begin{aligned}
&1 \in \mathbb{C} \\
&\downarrow \\
&1_V \in End(V) \cong V \otimes V^* \\
&\downarrow \\
&e_i \otimes e^i \in V \otimes V^* \\
&\downarrow T \\
&T(e_i) \otimes e^i \\
&= T_i^k e_k \otimes e^i \\
&\downarrow B_{V,V} \\
&T_i^k e^i \otimes e_k \\
&\downarrow \varepsilon_V \\
&T_i^k \delta_k^i = T_i^i
\end{aligned}
$$

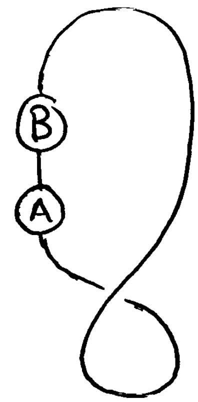
This shows that $tr(T)$ is indeed the usual trace of the matrix T.

tr(AB) = 

= 

= 

= 

= 

= tr(BA)