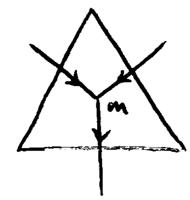
9 November 2004

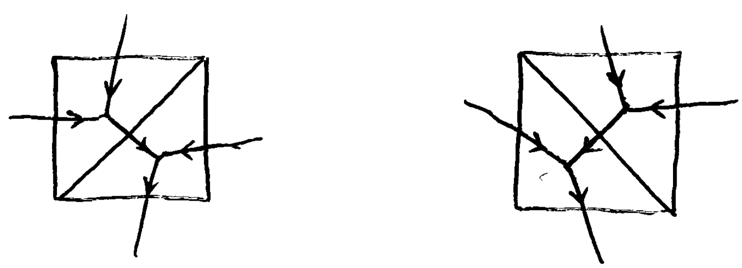
In our quest to build 2d TQFTs, we start with a vector space A equipped with a binary operation

$$m : A \otimes A \longrightarrow A$$

which we draw as

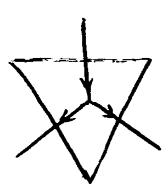


We demand that m be associative:



to get the 2-2 move.

To deal with triangles like this:



we want an isomorphism $A \cong A^*$, to get

$$A \cong A^* \xrightarrow{m^\dagger} A^* \otimes A^* \cong A \otimes A$$

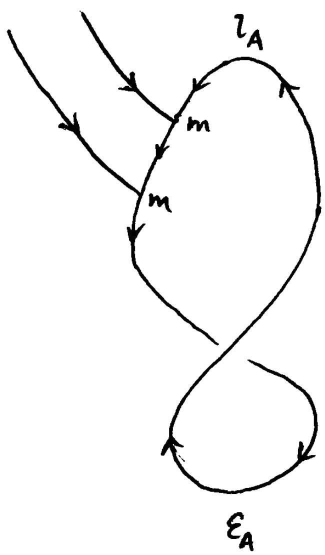
For this, we demanded that a certain God-given pairing

$$g : A \otimes A \longrightarrow \mathbb{C}$$

be nondegenerate, i.e. that

$$\begin{aligned} \# : A &\longrightarrow A^* \\ a &\longmapsto g(a \otimes -) \end{aligned}$$

be an isomorphism. What's this God-given g? It's



$$\begin{aligned} \iota_A : \mathbb{C} &\longrightarrow A \otimes A^* \cong \mathrm{End}(A) \\ \alpha &\longmapsto \alpha 1_A \end{aligned}$$

$$\begin{aligned} \varepsilon_A : A^* \otimes A &\longrightarrow \mathbb{C} \\ f \otimes v &\longmapsto f(v) \end{aligned}$$

In fact, to get a 2d TQFT all we need is this: a vector space A w. bilinear associative product m s.t. g is nondegenerate. Such a thing turns out to be a semisimple algebra!

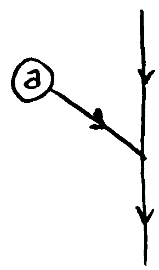
What's the meaning of g? We saw that given any linear operator $T : V \to V$,

$tr(T) =$ 

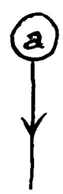
Note: left multiplication by $a \in A$ defines a linear operator:

$$L_a : A \longrightarrow A$$
$$b \longmapsto ab$$

which we draw as



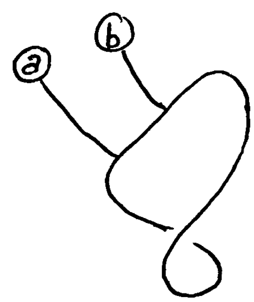
where



$$\begin{array}{c} \mathbb{C} \\ \big\downarrow \; 1 \mapsto a \\ A \end{array}$$

is our notation for an <u>element</u> of A.

So:

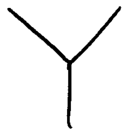
$$g(a \otimes b) =$$  $$= tr(L_b L_a) = tr(L_a L_b)$$

(This would be called the "Killing form" if A were a Lie algebra & $m = [\cdot,\cdot]$. A Lie algebra is called semisimple if g is nondegenerate.)

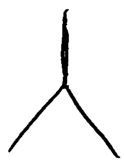
Since we're assuming g is nondegenerate, we have one isomorphism

$$\#: A \to A^*$$

which we'll use to identify A & A^*. So, we won't draw <u>arrows</u> on our string diagrams anymore! So:



$$m: A \otimes A \to A$$



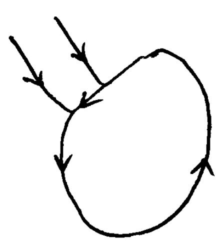
$$m^\dagger: A^* \to A^* \otimes A^*$$

gives

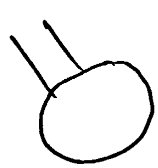
$$m^\dagger: A \to A \otimes A$$

Also, we'll define

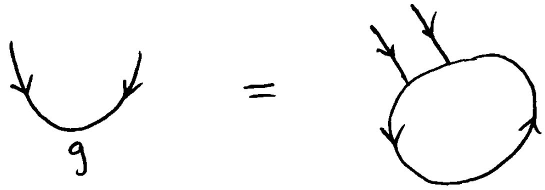
so that we can draw

$g =$ 

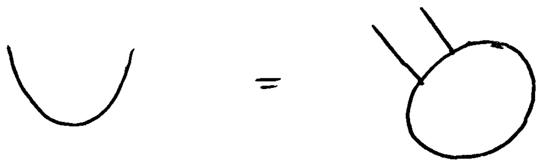
or just

$g =$ 

or:

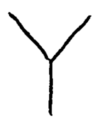


or:

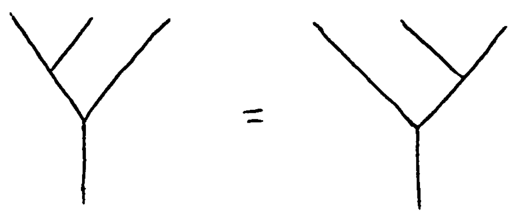


Don't worry – this can be proved to be harmless!

So: we have a vector space A with

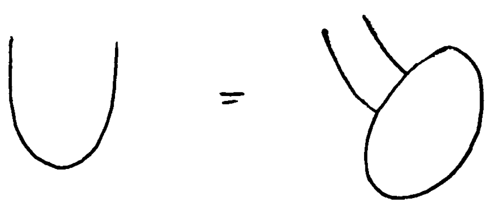


such that:



(associativity)

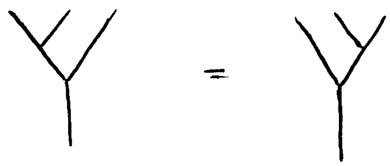
&



as well as the usual

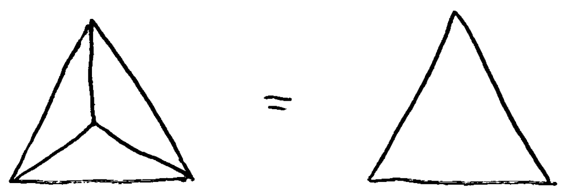
(nondegeneracy)

We saw that

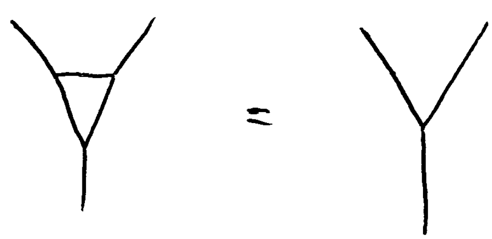


becomes the 2-2 move when we draw its Poincaré dual:

But what about the 1-3 move?

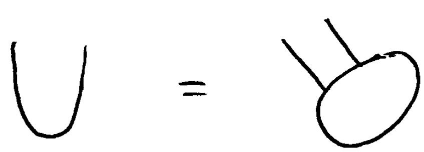


i.e

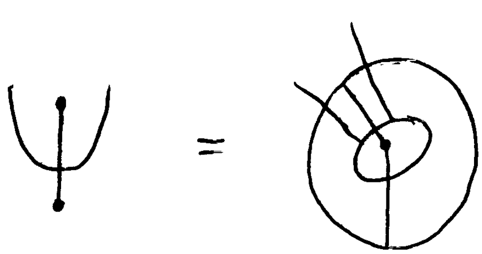


Claim: this <u>follows</u> from the rules we have!

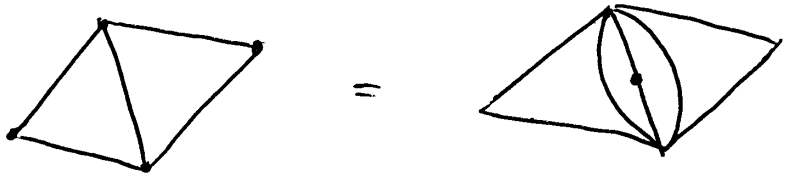
Let's draw the Poincaré dual of



i.e.

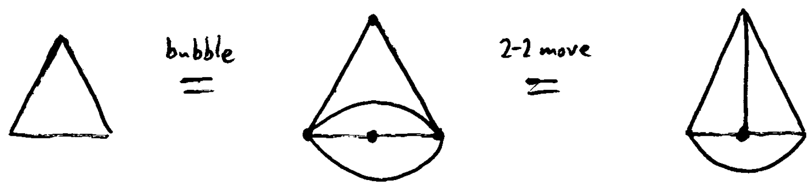


the <u>bubble move</u>, e.g.

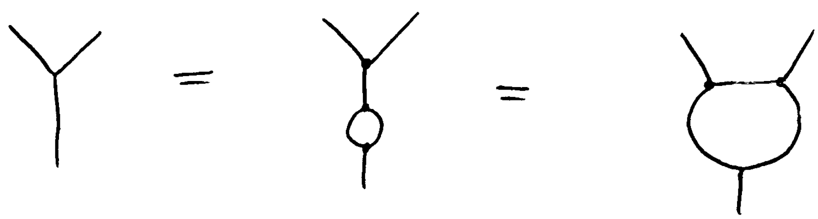


<u>Thm</u>: Given the 2-2 move, the 3-1 move is equivalent to the bubble move.

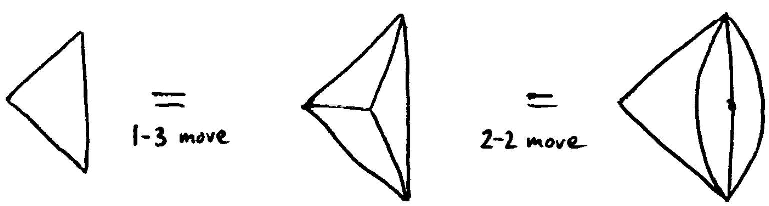
bubble $\Longrightarrow$ 3-1



or



3-1 $\Longrightarrow$ bubble



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