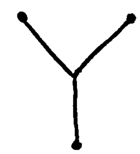
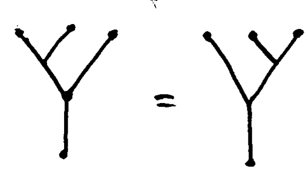
16 November 2004

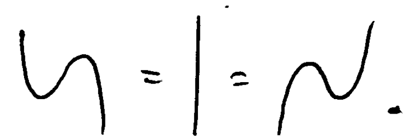
Given certain data we'll construct a 2d TQFT. These data consist of:

- A vector space A 
- A linear map $m : A \otimes A \to A$ called "multiplication" 

Such that:

- m is associative 
- m is semisimple: if we define $g : A \otimes A \to \mathbb{C}$ by ,

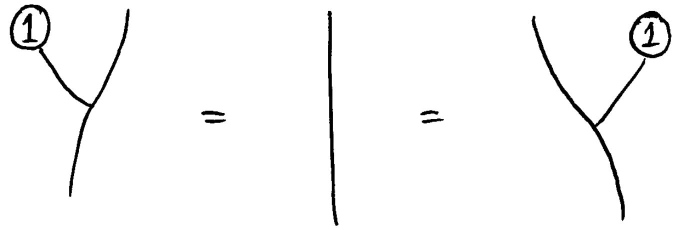
  then g is nondegenerate, i.e. $\exists\ \bar{g} : \mathbb{C} \to A \otimes A$ 

  such that .

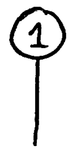
By the way, A is really an associative algebra, i.e. $\exists\, 1 \in A$ s.t. $1a = a = a1\ \forall a \in A$. Why? $1 \in A$ defines a linear operator $i : \mathbb{C} \to A$ via $i(1) = 1 \in A$, which we'd draw as:

$i =$ 

and want

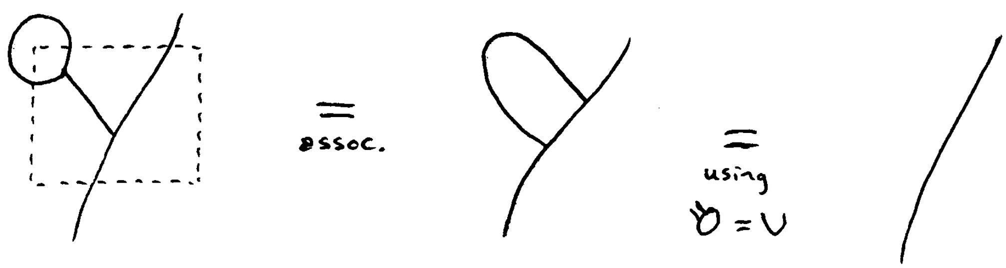


What is i, though? It's

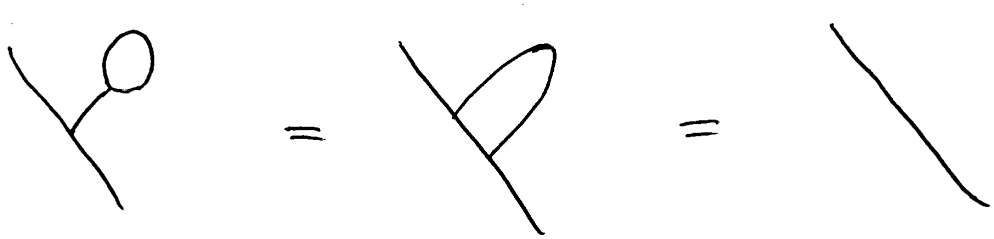
 :=  =

$$\mathbb{C} \xrightarrow{\bar{g}} A \otimes A \xrightarrow{m} A$$

Check:



and similarly



How do we get a 2d TQFT?

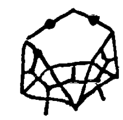
1) First define a "warmup"

$\tilde{Z}$ sending triangulated 1-manifolds to vector spaces

& sending triangulated cobordisms between 1-manifolds to linear operators

If S is a triangulated 1-manifold,

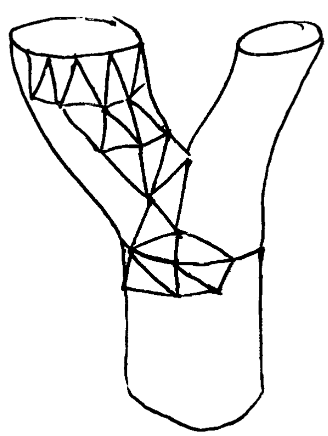
$\tilde{Z}(S) = A^{\otimes n}$ where $n = \#$ edges of S



If $M : S \to S'$ is a triangulated cobordism, $\tilde{Z}(M)$ is the linear operator obtained by turning M into a trivalent graph via Poincaré duality & treating that as a Feynman diagram via $Y = m$ $\cup = g$ $\cap = \bar{g}$.

2) Check $\tilde{Z}(MM') = \tilde{Z}(M)\tilde{Z}(M')$

M

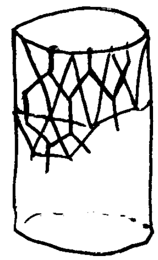


M'

Obvious

BUT: $\tilde{Z}$ does not preserve identities:

$$S \xrightarrow{1_S} S$$

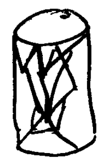


$S \times [0,1]$

need not give the identity operators.

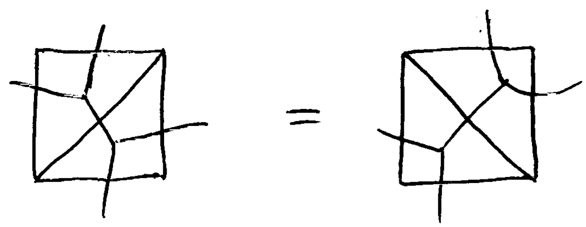
So need to fix this...

3) Check $\tilde{Z}(M)$ doesn't depend on the triangulation of $M - \partial M$:

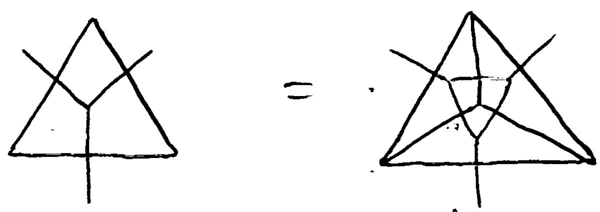
$\tilde{Z}\Big($  $\Big) = \tilde{Z}\Big($  $\Big)$

because we can go between any two of these triangulations using the Pachner moves and we've checked:

2-2



1-3



4) If we triangulate $S \times I$ in any way,

$$\tilde{Z}(S\times I)^2 = \tilde{Z}(S\times I)$$

i.e. $\tilde{Z}(S\times I)$, while not an identity, is idempotent (i.e a projection op.)

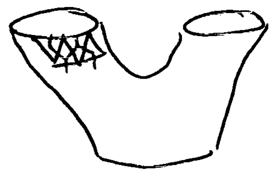
$$\tilde{Z}(S\times I)^2 \underset{\text{by } 2}{=} \tilde{Z}((S\times I)(S\times I))$$

$$\underset{\text{by } 3}{=} \tilde{Z}(S\times I)$$

18 Nov. 2004

To get a 2d TQFT:

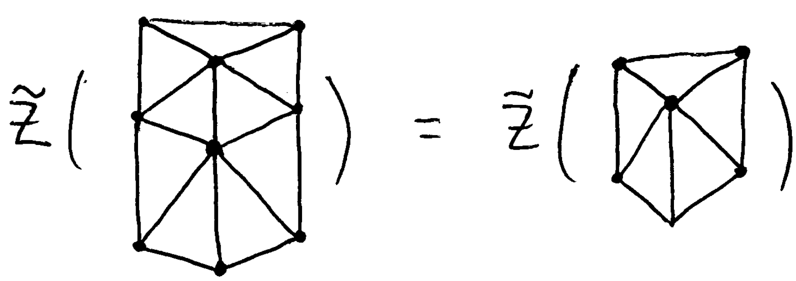
1) First define $\tilde{Z}(M): \tilde{Z}(S) \to \tilde{Z}(S')$ for <u>triangulated</u> cobordisms M between <u>triangulated</u> 1-manifolds S & S', using Feynman diagrams:



2) $\tilde{Z}(MM') = \tilde{Z}(M)\tilde{Z}(M')$

3) $\tilde{Z}(M)$ doesn't depend on triangulation of the interior of M; only on that of $\partial M = S \cup S'$.

4) $\tilde{Z}(S\times I)^2 = \tilde{Z}(S\times I)$ for any triangulation of $S\times I$ matching triangulation of S on $\partial(S\times I) = S \cup S$.



5) Next, define

$$Z(S) = \mathrm{Ran}\ \tilde{Z}(S\times I)$$

(still depends on triangulation of S).

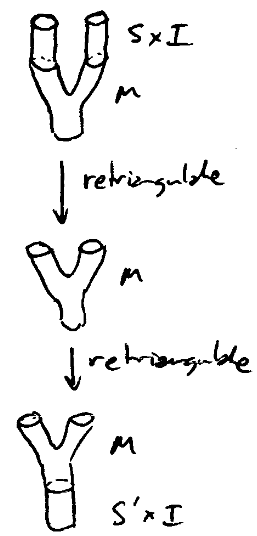
6) Check:

$$\tilde{Z}(M)\Big|_{Z(S)} : Z(S) \longrightarrow Z(S')$$

Here's how: $\tilde{Z}(M)Z(S) = Z$

Here's how:

$$\begin{aligned}\tilde{Z}(M)Z(S) &= \tilde{Z}(M)\ \mathrm{Ran}\ \tilde{Z}(S\times I)\\ &\subseteq \mathrm{Ran}\ \tilde{Z}(M)\tilde{Z}(S\times I)\\ &= \mathrm{Ran}\ \tilde{Z}(M(S\times I))\\ &= \mathrm{Ran}\ \tilde{Z}(M)\\ &= \mathrm{Ran}\ \tilde{Z}((S'\times I)M)\\ &\subseteq \mathrm{Ran}\ \tilde{Z}(S'\times I)\\ &= Z(S')\end{aligned}$$



7) Next define:

$$Z(M) = \tilde{Z}(M)\Big|_{Z(S)}$$

thus getting

$$Z(M): Z(S) \longrightarrow Z(S').$$

8) Check that Z is a functor from [triangulated 1-manifolds, triangulated cobordisms between these].

$$Z(MM') = Z(M)Z(M')$$

because

$$\tilde{Z}(MM') = \tilde{Z}(M)\tilde{Z}(M')$$

& Z is the restriction of $\tilde{Z}$ to a subspace. Also

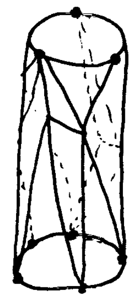
$$Z(S\times I) = \tilde{Z}(S\times I)\Big|_{\mathrm{Ran}\,\tilde{Z}(S\times I)} = 1\Big|_{\mathrm{Ran}\,\tilde{Z}(S\times I)} \quad \text{since } \tilde{Z}(S\times I) \text{ is a projection.}$$

$$= 1_{Z(S)} \qquad \text{by 4}$$

9) Check that given two triangulations of the same 1-manifold, say S_1 & S_2, that we have a specified isomorphism $\alpha : Z(S_1) \xrightarrow{\sim} Z(S_2)$.

For example:

$$S_1 \xrightarrow{M} S_2$$



Pick any way of triangulating a cylinder going from S_1 to S_2, say $M : S_1 \to S_2$, & let

$$\alpha = Z(M).$$

To see that α is an isomorphism, note its inverse is

$$\alpha^{-1} = Z(M^*)$$

where M^* is the time-reversed version of M. $\alpha\alpha^{-1}$ & $\alpha^{-1}\alpha$ are identity operators since MM^* & M^*M are triangulations of $S_1 \times I$ & $S_2 \times I$.

Using this trick we can convert Z into a functor:

$$Z : 2\mathrm{Cob} \longrightarrow \mathrm{Vect}$$

10) Check

$$Z : 2\mathrm{Cob} \to \mathrm{Vect}$$

is a symmetric monoidal functor, i.e. a TQFT

E.g. we want to specify an isomorphism

$$Z_{S_1,S_2} : Z(S)\otimes Z(S') \longrightarrow Z(S\cup S').$$

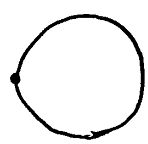
$$\begin{aligned}
Z(S_1)\otimes Z(S_2) &= \mathrm{Ran}\ \tilde{Z}(S_1\times I)\otimes \mathrm{Ran}\ \tilde{Z}(S_2\times I)\\
&\cong \mathrm{Ran}\ \tilde{Z}(S_1\times I)\otimes \tilde{Z}(S_2\times I)\\
&\cong \mathrm{Ran}\ \tilde{Z}(S\times I\cup S'\times I)\\
&\cong \mathrm{Ran}\ \tilde{Z}((S\cup S')\times I)\\
&\cong Z(S\cup S')
\end{aligned}$$

via a specified isomorphism.

via our Feynman diagram calculus

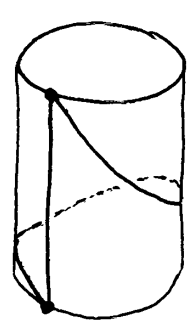
What is this TQFT like? We need to look at examples of semisimple algebras, e.g. those coming from groups, which give TQFTs called topological gauge theories.

But first, let's calculate $Z(S^1)$ in the TQFT coming from a semisimple algebra A. Choose a triangulation of S^1:

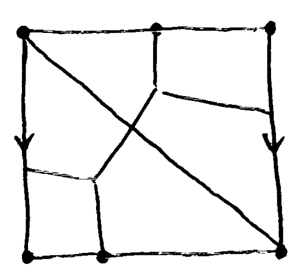


$\tilde{Z}(S^1)$ with this triangulation is A, since there's one edge.

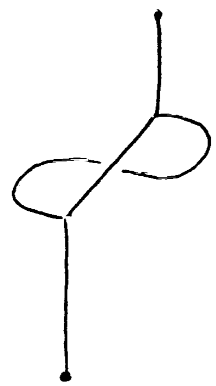
Choose a triangulation of $S^1\times I$



or unrolled:



The Poincare dual graph looks like



In fact the range of this is the <u>center</u> of A, i.e. the subspace of elts that commute with everything:

$$Z(S^1) = Z(A)$$

Zustandsumme ↗ ↖ Zentrum

Das Zentrum ist die Zustandsumme!