23 November 2004

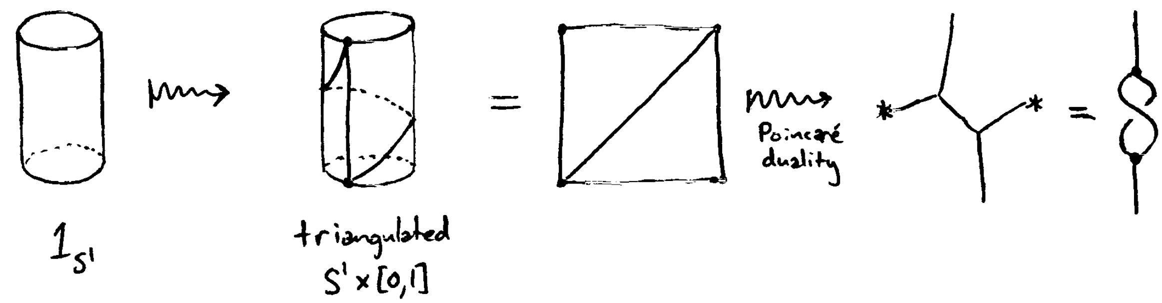
Given a semisimple associative algebra A, we built a 2d TQFT

$$Z : 2\mathrm{Cob} \longrightarrow \mathrm{Vect}$$

& we saw

$$Z(S^1) = \mathrm{Ran}\, p$$

where $p : A \to A$ has $p^2 = p$. This p is given by:



and we interpret

as

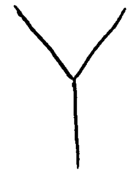
$$A \xrightarrow{m^\dagger} A \otimes A \xrightarrow{B_{A,A}} A \otimes A \xrightarrow{m} A \quad =: p \, .$$

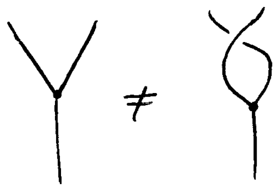
This satisfies $p^2 = p$ since the operator associated to $S^1 \times [0,1]$ is <u>triangulation independent</u>.

<u>Thm:</u> $\operatorname{Ran} p = Z(A)$, the center of the algebra A.

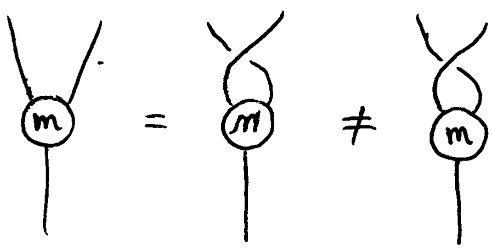
<u>Proof:</u> ① $Z(A) \subseteq \operatorname{Ran} p$, i.e. $a \in Z(A) \Rightarrow a \in \operatorname{Ran} p$ ($\Leftrightarrow a = pa$).

Notation:

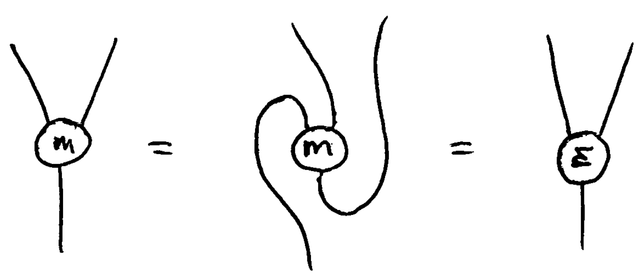
$a =$  $\mathbb{C} \xrightarrow{1 \mapsto a} A$ $\quad m =$  $A \otimes A \xrightarrow{m} A$

Note:  since A is not (necessarily) commutative,

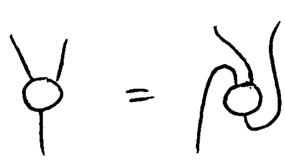
despite the topological temptation to "twist." To avoid this temptation, we can draw $m: A \otimes A \to A$ as a disc:

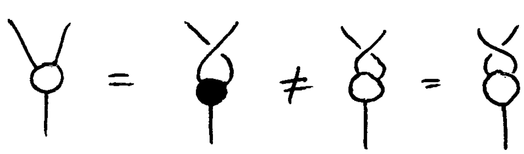


but:

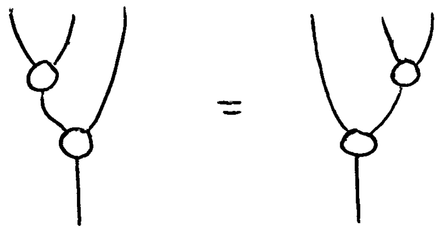


so it's even better to just color the front of the disc white and the back black



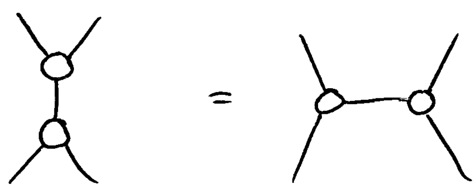


Then any topological manipulation of these diagrams in 4d is allowed, plus:



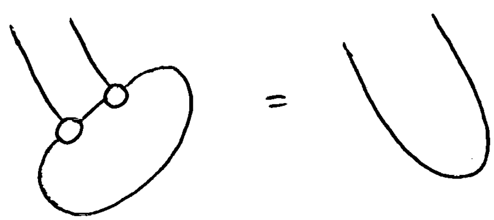
associative law

or



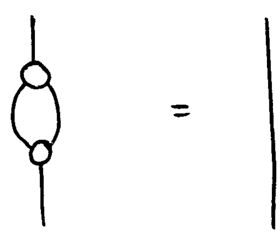
equivalent to associative law (the "I = H" law)

AND



semisimplicity

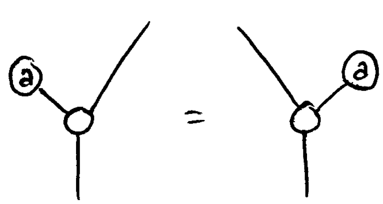
or



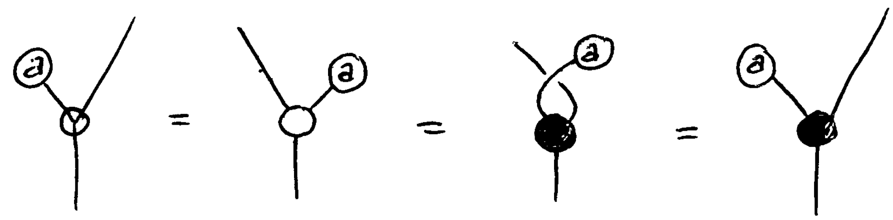
Now we consider

$p =$  and show $a \in Z(A) \Rightarrow pa = a$.

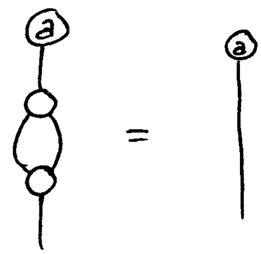
In pictures, $a \in Z(A)$ means:



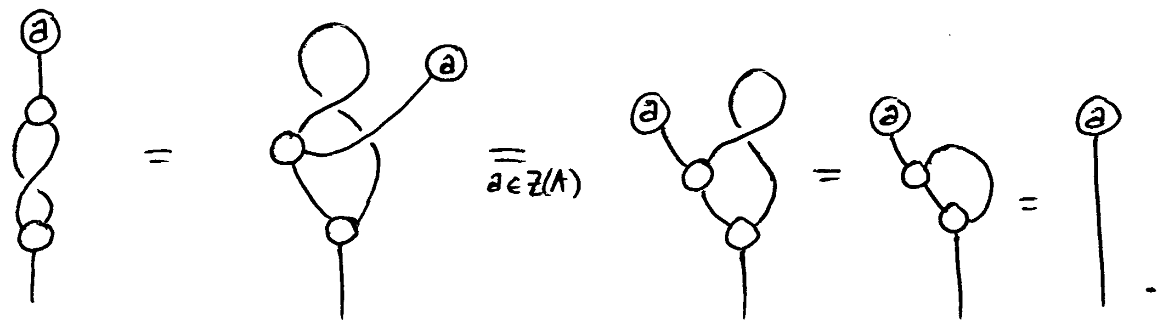
(i.e. $L_a = R_a$)

Or 

So we get

$pa =$  $= a$!

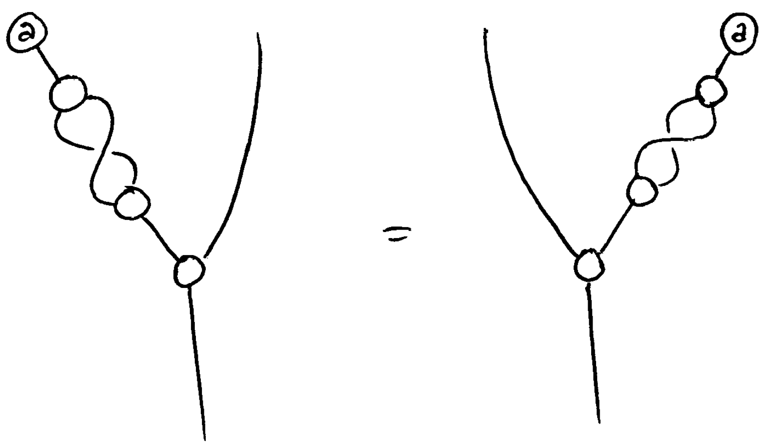
Alternate proof:



Conversely

② $\operatorname{Ran} p \leq Z(A)$, i.e. $pa \in Z(A) \quad \forall a \in A$.

In pictures this says



$L_{pa} \quad = \quad R_{pa}$

Here's how: