30 November 2004

We've seen that a semisimple algebra A gives us a 2d TQFT:

$$Z: 2\mathrm{Cob} \longrightarrow \mathrm{Vect}$$

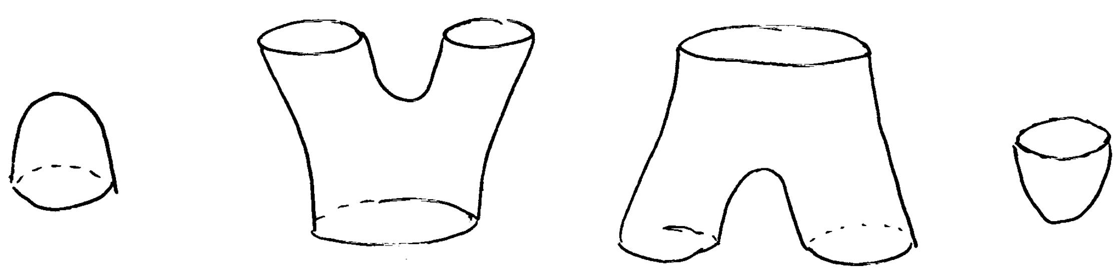
&

$$Z(S^1) = Z(A).$$

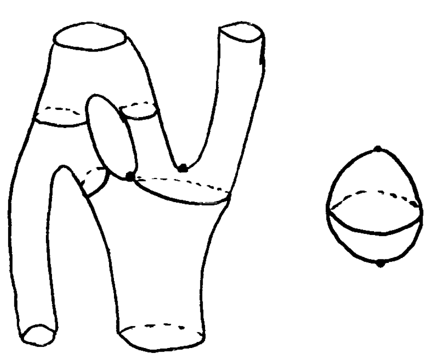
Any object of 2Cob (compact oriented manifold) is of the form $S^1 \cup S^1 \cup \cdots \cup S^1$ (up to isomorphism) so Z of it will be (isomorphic to)

$$Z(A) \otimes Z(A) \otimes \cdots \otimes Z(A)$$

so we know Z on objects. What about Z on morphisms? Every morphism in 2Cob can be built by composing and tensoring these "building blocks":

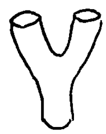


e.g.:



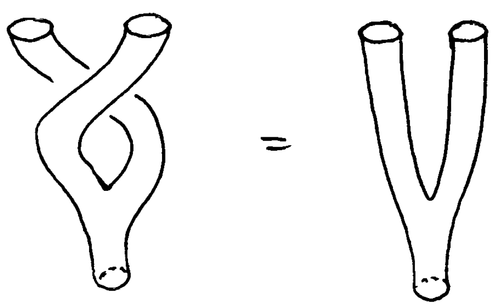
Each critical point gives a building block: saddle points give or , local maxima give and local minima give .
(Morse theory)

So Z will be completely determined by its values on these 4 building blocks. Let's guess:

$$Z\Big($$  $$\Big) = \begin{array}{c} Z(A)\otimes Z(A) \\ \big\downarrow {\scriptstyle M} \\ Z(A) \end{array}$$

where $M = m|_{Z(A)\otimes Z(A)}$, where m is multiplication in A.

Unlike m, M is commutative:

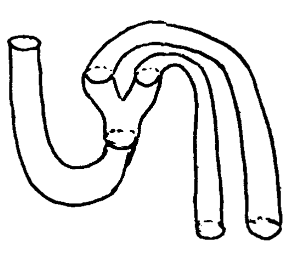
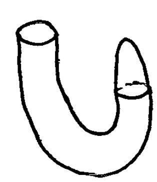


(whereas  $\neq$ )

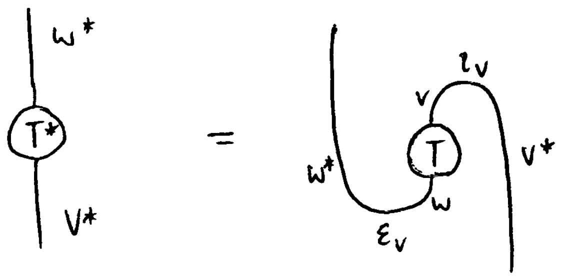
Similarly:

$$Z\Big($$  $$\Big) = \begin{array}{c} \mathbb{C} \\ \big\downarrow {\scriptstyle 1\mapsto I\in Z(A)} \\ Z(A) \end{array}$$

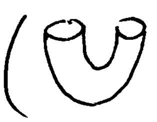
where $I = 1 \in A$. For , , let's use

 =     = 

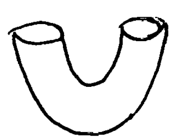
This is just like defining $T^*: W^* \to V^*$ given a linear operator $T: V \to W$:



Given this, Z will be determined once we know

$Z($  $)$ & $Z($  $)$

Let's guess:

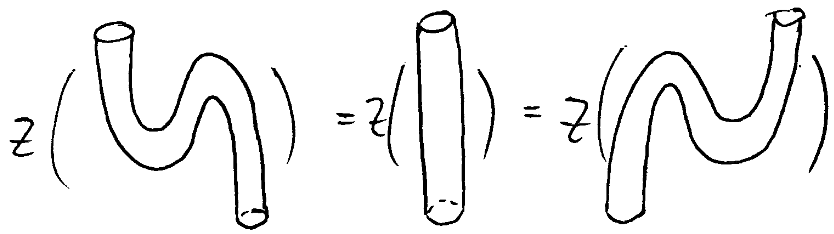
$Z($  $)$ =
$$\begin{array}{c} Z(A)\otimes Z(A) \\ \Big\downarrow G \\ \mathbb{C} \end{array}$$

where $G = g\big|_{Z(A)\otimes Z(A)}$ (Recall $g: A\otimes A \to \mathbb{C}$ was the nondegenerate pairing on A given by $\underset{g}{\cup} :=$)

Finally,

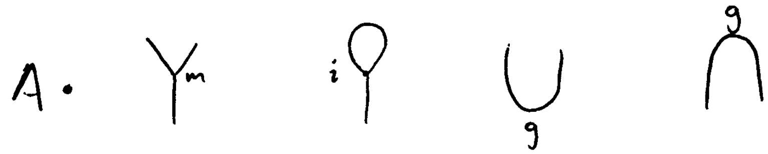
$Z($  $)$ =
$$\begin{array}{c} \mathbb{C} \\ \Big\downarrow \bar{G} \\ Z(A)\otimes Z(A) \end{array}$$

where $\bar{G}$ is determined in terms of G by insisting

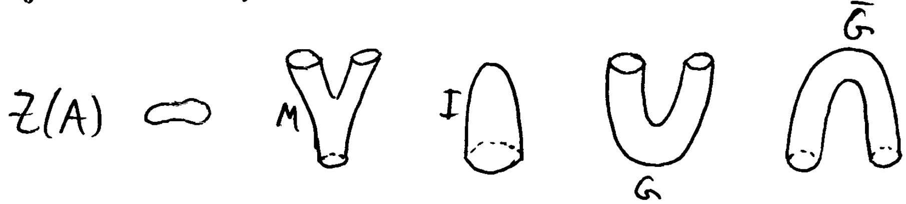


i.e. $(Z(A), Z(A), G, \bar{G})$ is an adjunction in Vect.

So starting from "particle physics":



we get "topological string theory":



GAUGE THEORY

We get a nice <u>example</u> of a semisimple algebra, & hence a 2d TQFT, from any finite group G. Namely, let $\mathbb{C}[G]$ be the vector space having G as basis:

$$\mathbb{C}[G] \ni a = \sum_{h \in G} a_h h \qquad a_h \in \mathbb{C}$$

This becomes an algebra by borrowing multiplication & unit from the group:

$$\underset{\substack{\uparrow \\ \text{mult. in } \mathbb{C}[G]}}{m}(h \otimes k) = \underset{\substack{\uparrow \\ \text{mult in } G}}{hk} \qquad \& \qquad \underset{\substack{\uparrow \\ \text{unit in } \mathbb{C}[G]}}{i}(1) = \underset{\substack{\uparrow \\ \text{unit in } G}}{1}$$

This is called the <u>group algebra</u> of G (Gives a functor $\mathrm{Grp} \to \mathrm{Alg}$)

<u>Thm:</u> For any finite group G, $\mathbb{C}[G]$ is semisimple.

<u>Proof:</u> Need to check that the pairing

$$g(a \otimes b) = \mathrm{tr}(L_a L_b)$$

is nondegenerate. We'll describe g as a matrix using a certain basis of $\mathbb{C}[G]$. Then g will be nondegenerate iff this matrix is invertible. We'll use G as the basis of $\mathbb{C}[G]$:

$$
\begin{aligned}
g_{hk} := g(h \otimes k) &= \mathrm{tr}(L_h L_k) = \\
&= \mathrm{tr}(L_{hk}) \\
&= \sum_{\ell \in G} \langle \ell, L_{hk} \ell \rangle \\
&= \sum_{\ell \in G} \langle \ell, hk\ell \rangle \\
&= \sum_{\ell \in G} \delta_{\ell, hk\ell} \\
&= \sum_{\ell \in G} \delta_{1, hk} \\
&= |G| \, \delta_{1, hk} \\
&= |G| \, \delta_{h, k^{-1}}
\end{aligned}
$$

inner prod. on $\mathbb{C}[G]$ s.t. group elts ℓ form an orthonormal basis

this is a diagonal entry in the matrix for L_{hk}

this is why we need $|G|$ finite — otherwise not trace class.

This matrix $g_{hk} = |G|\,\delta_{hk^{-1}}$ looks like

$$|G| \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & & & 0 & & & \\ \vdots & 0 & & & 0 & & \vdots & \\ & 0 & & & 1 & & & \\ \vdots & 0 & & & 0 & & \ddots & \\ & 0 & & & 0 & & & \\ & 0 & & & 0 & & & \\ & 0 & & & 0 & & & \end{pmatrix}$$

(i.e. one 1 in each row & each column — it's a permutation matrix, so its determinant is $\pm|G|^{|G|}$, in particular its det is nonzero so g is nondegenerate. ∎

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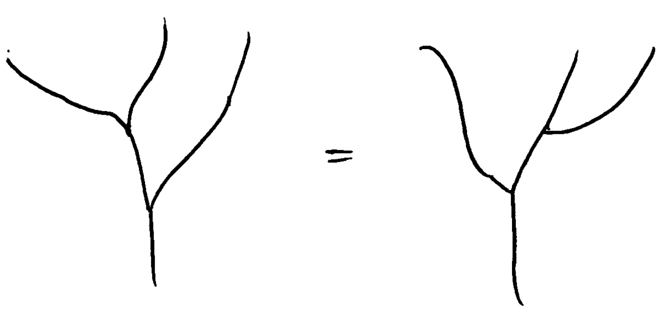
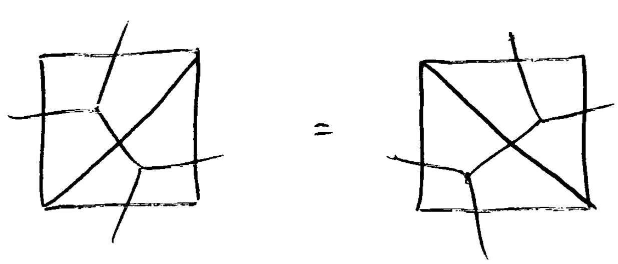
2 Dec 2004

A sneak preview of next quarter...

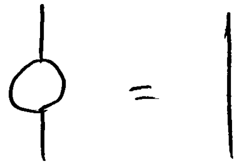
So far we've constructed 2d TQFTs from certain nice monoids, namely semisimple algebras. A <u>monoid</u> has an associative multiplication:



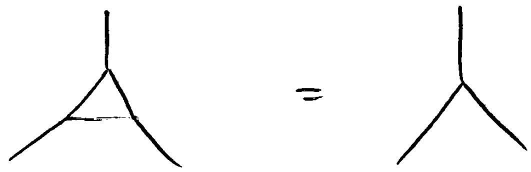
which gives the 2-2 Pachner move:



Being a semisimple algebra gives



which together with associativity gives

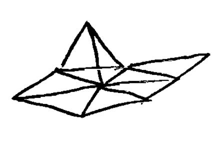


which is the 1-3 Pachner move:

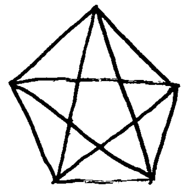
In 3 dimensions, there are again 2 Pachner moves. In 2d, the Pachner moves are pictures of the front and back of a tetrahedron:

2-2 1-3

To change a 2d triangulation, glue a tetrahedron on top of it:



So, following our collective nose, we can guess that in 3d, the Pachner moves will be pictures of the front and back of a 4-simplex:

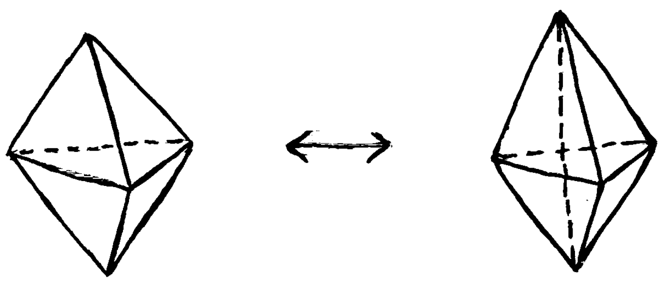


a 4-simplex has 5 tetrahedral faces

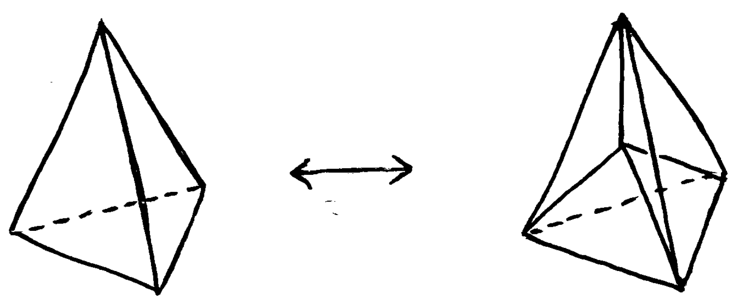
$2+3 = 1+4 = 5$

There's a "2-3" and a "1-4" Pachner move

2-3



1-4



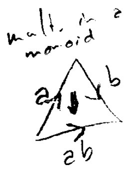
which suffice to go between any two triangulations of a compact manifold.

To build 3d TQFTs we want to calculate operators from any triangulated cobordisms:

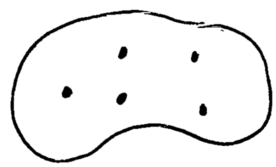
$$\tilde{Z}(M) : \tilde{Z}(S) \longrightarrow \tilde{Z}(S')$$

which is invariant under Pachner moves on M. We need some generalization of a semisimple algebra. What comes after "monoid"?

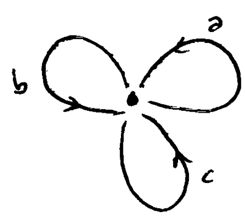
$\bullet$	$\longrightarrow$	$\Downarrow$	
sets	categories	2-categories	...
monoids	monoidal categories	monoidal 2-categories.	...
commutative monoids	braided monoidal categories	braided monoidal 2-categories	...
⋮	⋮	⋮	



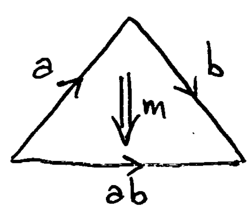
A monoid seems like a set:



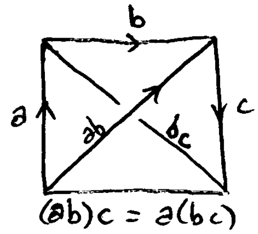
but it's really a category with one object



so the multiplication operator looks 2-dimensional



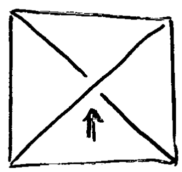
& the associative law looks 3-dimensional:



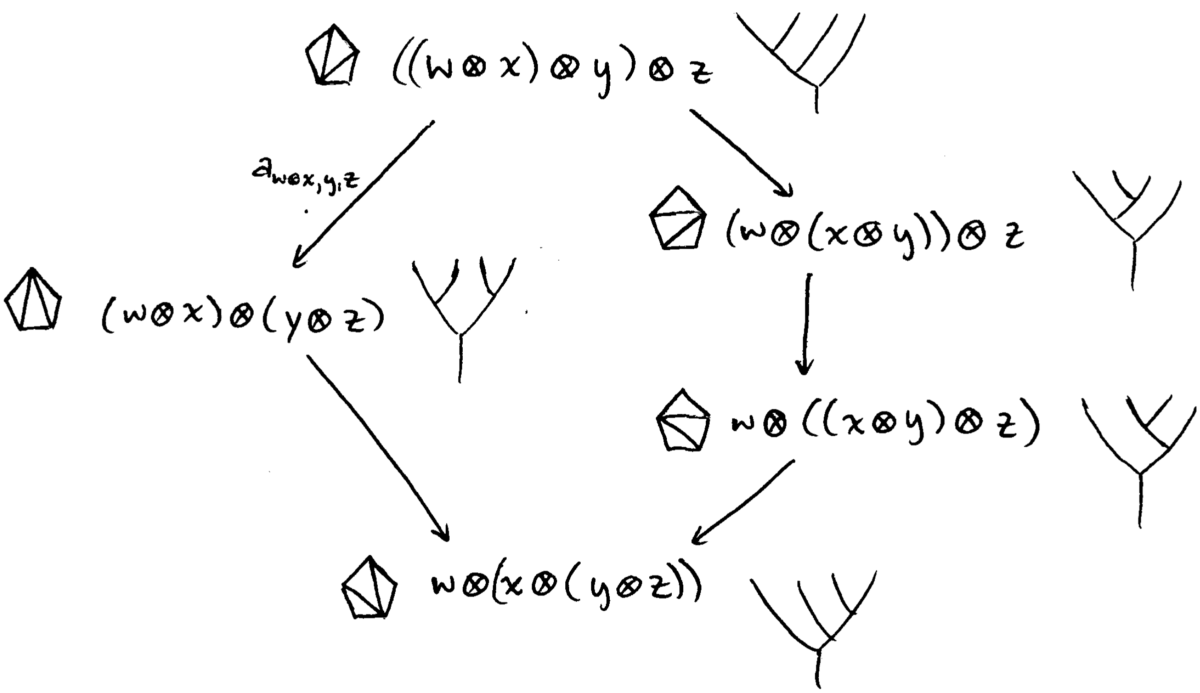
To repeat this stunt one dimension up, we'll use nice monoidal categories & see that their definition contains one of the Pachner moves in disguise. In a monoidal category, we have an associator:

$$a_{x,y,z} : (x \otimes y) \otimes z \xrightarrow{\sim} x \otimes (y \otimes z)$$

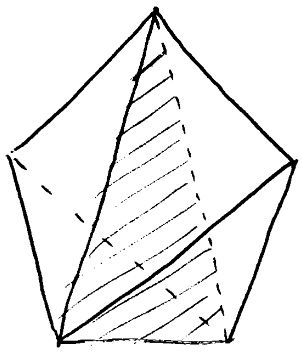
which is the <u>process</u> of going from the front to the back of a tetrahedron



which satisfies the pentagon identity:

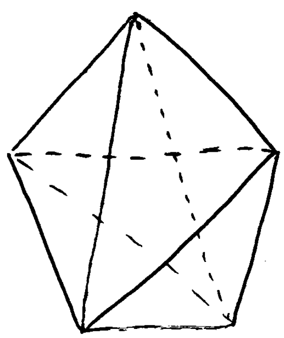


Secretly this is a picture of the front & back of a 4-simplex: the front (left) has 2 tetrahedra; the back (right) has 3!



2 tetrahedra
sharing one triangular face.

||



3 tetrahedra
sharing one edge

So the pentagon identity is the 2-3 move!

But, to get the 1-4 move, we'll need not just any monoidal category, but a "semisimple 2-algebra." What's this? An algebra is a monoid in Vect. Similarly a 2-algebra should be a monoidal category in 2-Vect. What's a 2-vector space?

$\mathbb{C}$ is the set of all numbers;
a list of numbers is a vector, so a vector space looks like $\mathbb{C}^n$.

Vect is the category of all vector spaces
a list of vector spaces is a 2-vector
so a 2-vector space looks like Vect^n

Alissa & John have a kind of different categorification of Vect.

2-Vect is the 2-category of all 2-vector spaces;
a list of 2-vector spaces is a 3-vector
so a 3-vector space looks like $(2\text{-Vect})^n$ ...

Our favorite kind of semisimple algebra for 2d TQFTs is $\mathbb{C}[G]$
" " 2-algebra for 3d " " $\mathrm{Vect}[G]$

An elt of $\mathbb{C}[G]$ looks like $\sum_{g\in G} a_g g$ with $a_g \in \mathbb{C}$
An object of $\mathrm{Vect}[G]$ looks like $\bigoplus_{g\in G} a_g g$ with $a_g \in \mathrm{Vect}$

(Note: $\mathrm{Vect}[G]$ can also be thought of as vector bundles over G)