

Nondegenerate Pairings

John C. Baez, November 9, 2004

First let's straighten out something that was explained rather poorly in class....

Let Vect be the category of finite-dimensional complex vector spaces. If $V \in \text{Vect}$, we all know that a linear map

$$g: V \otimes V \rightarrow \mathbb{C}$$

yields a linear map

$$\sharp: V \rightarrow V^*$$

as follows:

$$\sharp(v)(w) = g(v \otimes w)$$

for all $v, w \in V$. But let's see how to do this using string diagrams! For this, we might as well work quite generally. So, assume C is any monoidal category and (x, x^*, i_x, e_x) is any adjunction in C .

1. Given a morphism

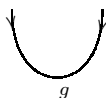
$$g: x \otimes x \rightarrow 1$$

in C , called a **pairing**, construct a morphism

$$\sharp: x \rightarrow x^*$$

using g and the adjunction (x, x^*, i_x, e_x) . Make sure that in the special case $C = \text{Vect}$, this morphism \sharp reduces to the linear operator given by the previous formula.

2. If the string diagram for g looks like this:



what does the string diagram for \sharp look like? (*Hint: if you have trouble drawing pictures in LaTeX, you can just copy and modify some pictures from the LaTeX file of this or the previous homework assignment, available on my website. Just remember to include the line `\usepackage[all]{xy}`.*)

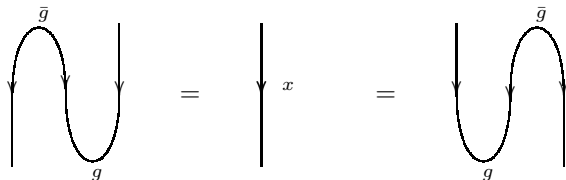
In this general situation, we say the pairing g is **nondegenerate** if \sharp is an isomorphism. We then denote the inverse of \sharp by

$$\flat: x^* \rightarrow x$$

3. If $g: x \otimes x \rightarrow 1$ is nondegenerate, use the morphism $\flat: x^* \rightarrow x$ and whatever else we have at hand to construct a morphism

$$\bar{g}: 1 \rightarrow x \otimes x$$

such that (x, x, \bar{g}, g) is an adjunction. In other words, find \bar{g} such that these identities hold:



There, now that's straightened out....

4. Suppose $V \in \text{Vect}$, and let $A = \text{End}(V)$ be the vector space of linear transformations of V , made into an algebra in the usual way. Define the pairing

$$g: A \otimes A \rightarrow \mathbb{C}$$

by

$$g(a \otimes b) = \text{tr}(L_a L_b)$$

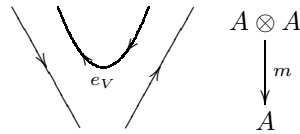
where L_a is left multiplication by $a \in A$:

$$\begin{aligned} L_a: A &\rightarrow A \\ b &\rightarrow ab \end{aligned}$$

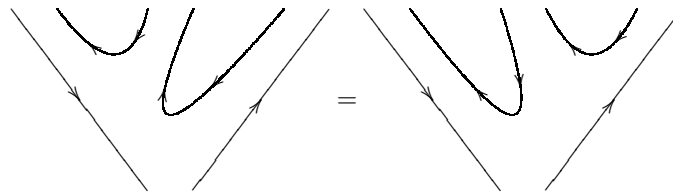
Show that g is nondegenerate.

Any algebra has a pairing of the above form; if the pairing is nondegenerate the algebra is **semisimple**. This is either a definition or a theorem depending on your taste: if we define a semisimple algebra to be a direct sum of algebras with no nontrivial two-sided ideals, it's a theorem. The algebras $\text{End}(V)$ are usually called **matrix algebras**, since $\text{End}(\mathbb{C}^n)$ is the algebra of $n \times n$ complex matrices. So, we're seeing that matrix algebras are semisimple. A harder theorem is that any semisimple algebra is a direct sum of matrix algebras!

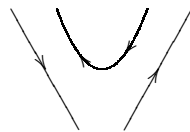
By the way, if we think of $A = \text{End}(V)$ as $V \otimes V^*$, we can express matrix multiplication in terms of string diagrams as follows:



Then there's an easy diagram proof that matrix multiplication is associative:



In physics we sometimes use V as the Hilbert space of states for a particle and V^* as the space of states for its corresponding antiparticle. If this particle happens to be a quark, $V \otimes V^*$ is the Hilbert space for a quark-antiquark pair, otherwise known as a **meson**. Then this:



is the Feynman diagram for an interaction where two mesons collide and become one. A process like this indeed occurs in nature... should we call it 'meson multiplication'?

That would be cute, but it's not precisely right. When two mesons collide to become one, the relevant map is usually not matrix multiplication $m: a \otimes b \mapsto ab$, but instead the commutator $[\cdot, \cdot]: a \otimes b \mapsto ab - ba$. Thus, mesons naturally form not an associative algebra, but a Lie algebra! For example, in Gell-Mann's 'Eightfold Way' theory there are eight mesons forming a basis of the Lie algebra $\mathfrak{su}(3) \subset \mathbb{C}^3 \otimes \mathbb{C}^{3*}$. Later, people copied this idea to describe other force-carrying particles. For example, in quantum chromodynamics there are eight gluons forming a basis of $\mathfrak{su}(3)$, and the Lie bracket in $\mathfrak{su}(3)$ describes an interaction where two gluons become one!