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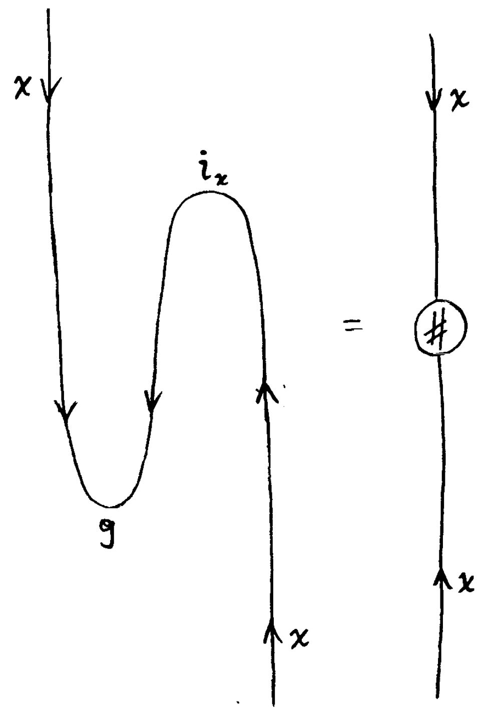
QUANTUM GRAVITY SEMINAR
HW#5 — Nondegenerate Pairings

1. & 2.)

$$\begin{tikzcd}
x \arrow[d, "r_x^{-1}"] \arrow[ddddd, bend right=60, "\#"'] \\
x \otimes 1 \arrow[d, "1_x \otimes i_x"] \\
x \otimes (x \otimes x^*) \arrow[d, "a^{-1}_{x,x,x^*}"] \\
(x \otimes x) \otimes x^* \arrow[d, "g \otimes 1_{x^*}"] \\
1 \otimes x^* \arrow[d, "\ell_{x^*}"] \\
x^*
\end{tikzcd}$$



In the case of Vect, the monoidal category of finite dim. complex vector spaces, with $g: V \otimes V \to \mathbb{C}$ any linear map, this becomes (see previous homework):

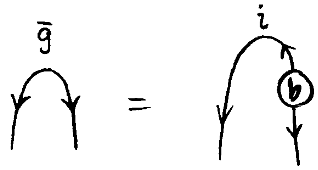
$$\begin{tikzcd}
V \arrow[d, "r_V^{-1}"] & v = v^i e_i \arrow[d, mapsto] \\
V \otimes \mathbb{C} \arrow[d, "1 \otimes i_V"] & v \otimes 1 \arrow[d, mapsto] \\
V \otimes (V \otimes V^*) \arrow[d, "\text{assoc.}"] & v \otimes (e_i \otimes e^i) \arrow[d, mapsto] \\
(V \otimes V) \otimes V^* \arrow[d, "g \otimes 1_{V^*}"] & (v \otimes e_i) \otimes e^i \arrow[d, mapsto] \\
\mathbb{C} \otimes V^* \arrow[d, "\ell_{V^*}"] & g(v \otimes e_i) \otimes e^i \arrow[d, mapsto] \\
V^* & g(v \otimes e_i) e^i
\end{tikzcd}$$

where $e_1, \ldots, e_n$ is a basis for V.

So:

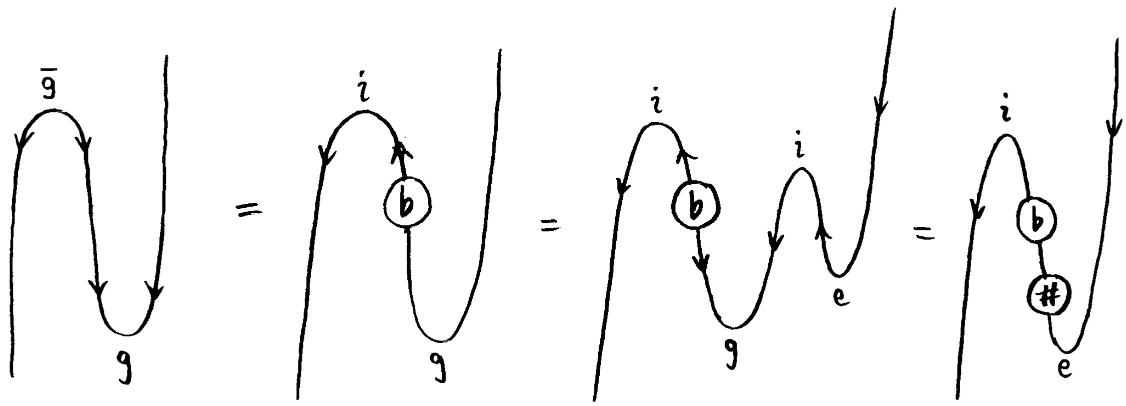
$$\begin{aligned}
v^\#(w) &= g(v \otimes e_i) e^i (w^j e_j) \\
&= g(v \otimes e_i) w^i \\
&= g(w^i v \otimes e_i) \\
&= g(v \otimes w^i e_i) \\
&= g(v \otimes w).
\end{aligned}$$

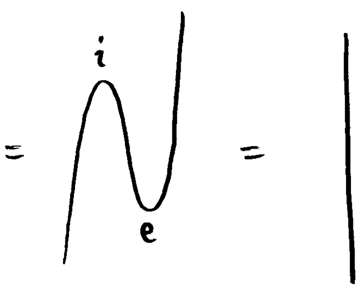
3.) If g is nondegenerate, let $\bar{g}: 1 \to x \otimes x$ be defined by



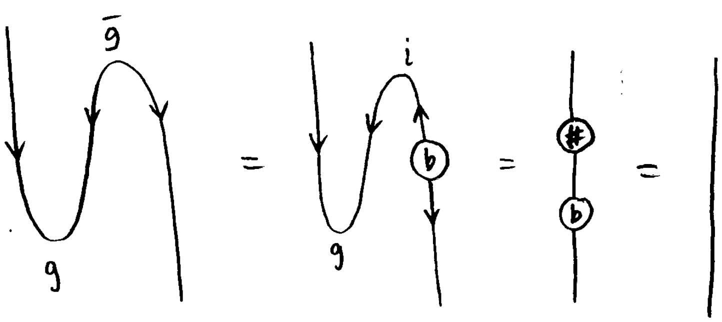
where $b = \#^{-1} : x^* \to x$.

Then



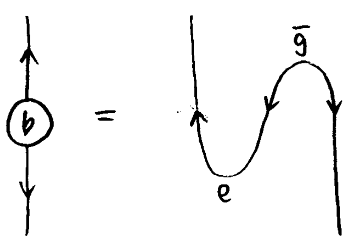


, and also

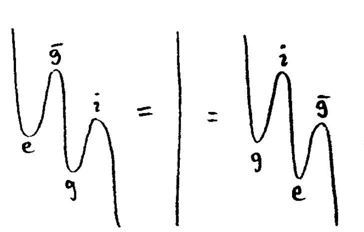


. So $(x, x, \bar{g}, g)$ is an adjunction.

(Note: I constructed $\bar{g}$ given the existence of b. Sometimes, however, it's nice to have an expression for b in terms of $\bar{g}$:



since



.)

4.) $V \in \mathrm{Vect}$, $A = \mathrm{End}(V)$. Let $g: A \otimes A \to \mathbb{C}$ be defined by $g(a \otimes b) = tr(ab)$. To see that $\#: A \to A^*$ given by $a^\#(b) = tr(ab)$ is an isomorphism, we need only check that it is injective, since A & A^* have the same (finite) dimension. But if $a^\# = 0 \in A^*$ for some $a \in A$, then we have

$$
\begin{aligned}
0 = a^\#(b) &= tr(ab) \qquad \forall b \in A \\
&= a^i_j b^j_i
\end{aligned}
$$

But we may choose the numbers b^j_i arbitrarily. In particular, letting b be an elementary matrix with a 1 in the (k,ℓ)-entry and zeros elsewhere shows that the (ℓ,k)-entry of a is zero. Since k,ℓ are arbitrary, this shows a is the zero matrix. So $\ker(\#) = 0$ and $\therefore$ g is nondegenerate.

(Note: thinking of A as $\mathrm{Mat}_{\dim V}\mathbb{C}$, it makes sense to write just $tr(a)$ for $tr(L_a)$ for any $a \in A$.)