QUANTUM GRAVITY SEMINAR
Fall 2006

QUANTIZATION & COHOMOLOGY

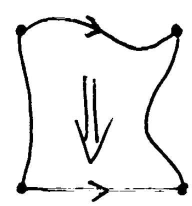
3 October 2006

particle statics

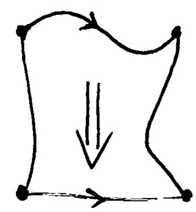
particle dynamics

string statics

string dynamics



2-brane statics



2-brane dynamics

... and so on:

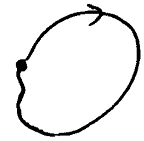
p-brane dynamics can be reinterpreted as (p+1)-brane statics...

Here we're taking a space X & forming

$$PX = \{ \gamma : [t_0, t_1] \to X \}$$

and then PPX, $PPPX$, etc. PX is the "configuration space for open strings in X" but it's also good to look at the "configuration space for closed strings in X" — i.e. loops rather than paths:

$$LX = \{ \gamma \in PX : \gamma(t_0) = \gamma(t_1) \}$$



Similarly, we can form LLX, $LLLX$, ...

These spaces are related to the cohomology of X in a certain way, since the first cohomology group of a topological space X can be described as

$$[X, U(1)]$$

— the set of homotopy classes of maps from X to

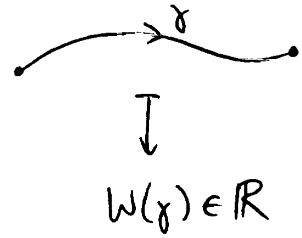
$$U(1) = \{ z \in \mathbb{C} : |z| = 1 \}.$$

What do maps $S : X \to U(1)$ have to do with loops $\gamma : U(1) \to X$?! This takes a while to explain, but let's see how such maps $S : X \to U(1)$ show up in physics of particles, & then how maps $S : LX \to U(1)$ show up in physics of closed strings, etc,.. & are related to higher cohomology gps!

<u>Classical Statics of point particles</u> (and really general classical systems)

We have a <u>configuration space</u> X whose points $x \in X$ are possible positions for our particle (or the position of our general classical system). Often X is a manifold and we choose a 1-form F on X called the <u>force field</u>. The <u>work</u> done on the particle as it moves along a path $\gamma : [t_0, t_1] \to X$ is defined to be

$$W(\gamma) = \int_\gamma F$$



A position $x \in X$ is an equilibrium if

$$F(x) = 0$$

i.e. for any infinitesimal displacement $v \in T_x X$ we have

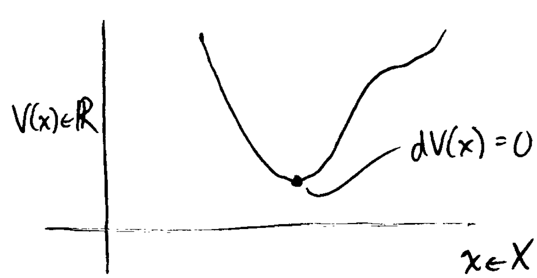
$$\underbrace{F(x)}_{\in T_x^* X}(v) = 0$$

or in D'Alembert's terminology, the virtual work $F(x)(v)$ vanishes.

Often

$$F = -dV$$

for some 0-form, or function, $V : X \to \mathbb{R}$ called the <u>potential energy</u>



In this case, any critical point of V is an equilibrium:

$$dV(x) = 0$$

This fact is often called the principle of least energy (since often x is a minimum). Also:

$$W(\gamma) = V(\gamma(t_0)) - V(\gamma(t_1))$$

by the fundamental theorem of calculus.

Classical dynamics of point particles

Morally, this is all the same except instead of using X we use PX or more precisely:

$$P_{x_0,x_1}X = \{ \gamma \in PX : \gamma(t_i) = x_i \quad i = 0,1 \}$$

The idea now is that the particle chooses a path from x_0 to x_1 such that

$$dS(\gamma) = 0$$

where

$$S: P_{x_0,x_1}X \longrightarrow \mathbb{R}$$

is called the action. The equation $dS(\gamma) = 0$ is called the principle of least action (since often γ is a minimum).

Classical statics of a string

This is very much like classical dynamics of a particle, reinterpreting $P_{x_0,x_1}X$ as the configuration space of an open string & S as a potential. But note – the string's ends are nailed down at x_0 & x_1. We might want to drop that constraint & use PX or, for closed strings, LX, or

$$\Omega X = \{ \gamma \in LX : \gamma(t_0) = \gamma(t_1) = * \}$$

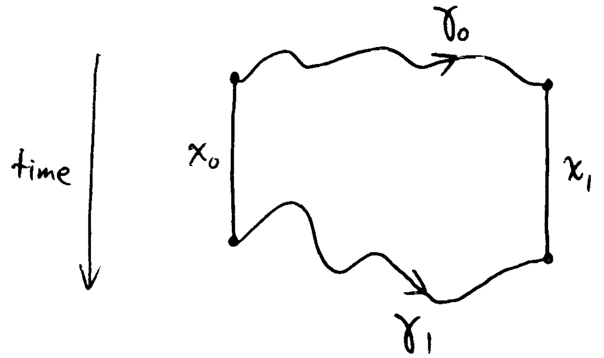
for some basepoint $* \in X$.

Classical dynamics of strings.

Just like classical dynamics of particles, but replacing X by $P_{x_0, x_1} X$, PX, LX, or ΩX (depending on the type of string) So we need an action

$$S : P_{\gamma_0, \gamma_1} P_{x_0, x_1} X \longrightarrow \mathbb{R}$$

if we're studying strings with ends nailed down



We want to find worldsheet(s)

$$\Sigma \in P_{\gamma_0, \gamma_1} P_{x_0, x_1} X \qquad \Sigma : [t_0, t_1] \times [t_0', t_1'] \longrightarrow X$$

with

$$dS(\Sigma) = 0$$

And so on...

More recently, people have studied quantum versions of all these ideas. This started explicitly with Feynman, who said:

the <u>quantum</u> dynamics of particles is also governed by

$$S: P_{x_0,x_1}X \longrightarrow \mathbb{R}$$

but in a new way! Instead of choosing the path γ with $dS(\gamma) = 0$, it chooses <u>all</u> paths with certain "amplitudes" given by

$$e^{iS(\gamma)/\hbar} \in U(1)$$

where $\hbar$ is Planck's constant, with units of action (often we'll set $\hbar = 1$).

This is how $U(1)$ gets into the game. In retrospect, we can do classical dynamics of particles using not S but $e^{iS} = A$, since

$$dA(\gamma) = 0$$

makes sense (given $A: P_{x_0,x_1}X \longrightarrow U(1)$ we can define a complex-valued 1-form dA on $P_{x_0,x_1}X$) & it's equivalent to

$$dS(\gamma) = 0$$

Similarly we can do classical statics of a particle using not

$$V: X \longrightarrow \mathbb{R}$$

but

$$e^{iV} : X \to U(1)$$

So, in short we have:

statics of particles	$A : X \longrightarrow U(1)$	$p=0$
dynamics of particles $\parallel$ statics of strings	$A : PX \longrightarrow U(1)$	$p=1$
$\vdots$		
dynamics of p-branes $\parallel$ statics of (p+1)-branes	$A : P^p X \longrightarrow U(1)$	