5 October 2006

In the untyped λ-calculus, every expression represents a "function" or "operation" which eats any other operation & spits out other operations. This is bizarre & incestuous, so we should introduce the typed λ-calculus, where each operation acts on things of a certain type & spits out things of some specified type.

What's a "type", really? And what's an "operation"? In the modern view, we fix any category $\mathcal{T}$ & call its objects types and morphisms operations

$$f: \overset{\text{source}}{X} \longrightarrow \overset{\text{target}}{Y}$$

A functor

$$F: \mathcal{T} \longrightarrow \mathrm{Set}$$

assigns a set $F(A)$ to each type $A \in \mathcal{T}$ & an actual function $F(f): F(A) \longrightarrow F(B)$ for each operation $f: A \to B$.

Or, we could replace Set by some other category $\mathcal{C}$. Then we think of the functor $F: \mathcal{T} \to \mathcal{C}$ as a <u>model</u> of the <u>theory</u> $\mathcal{T}$ in the <u>context</u> $\mathcal{C}$.

Suppose

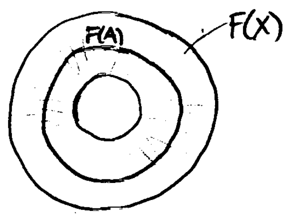
$$\mathcal{T} = \left\{ \begin{tikzcd} A \arrow[r, bend left, "i"] & X \arrow[l, bend left, "r"] \end{tikzcd} \quad : \quad r \circ i = 1_A \right\}$$

(we get $1_A, 1_X, i \circ r$ as well, but no more)

What's a model $F : \mathcal{T} \longrightarrow \mathrm{Top}$ called?

$$\begin{tikzcd} F(A) \arrow[r, bend left, "F(i)"] & F(X) \arrow[l, bend left, "F(r)"] \end{tikzcd} \qquad F(r) \circ F(i) = 1_{F(A)}$$

This is called a retraction



e.g. the circle is a retract of the annulus.

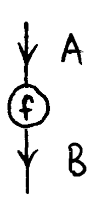
What do you call a model of $\mathcal{T}$ in AbGp?

$$0 \longrightarrow \ker F(r) \longrightarrow F(A) \underset{F(r)}{\overset{F(i)}{\leftrightarrows}} F(B) \longrightarrow 0$$

(surjection

It's called a split exact sequence.

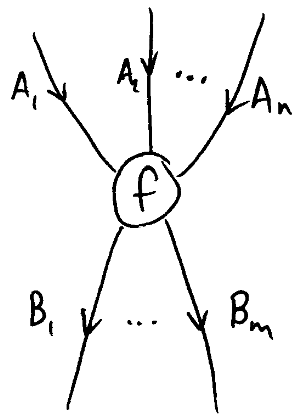
We describe things (types) with operations satisfying equations this way. Alas, these operations have just <u>one</u> input and one output:



To allow many inputs & many outputs we need monoidal categories — i.e. categories with a "tensor product" so we have morphisms like

$$f : A_1 \otimes \cdots \otimes A_n \longrightarrow B_1 \otimes \cdots \otimes B_m$$

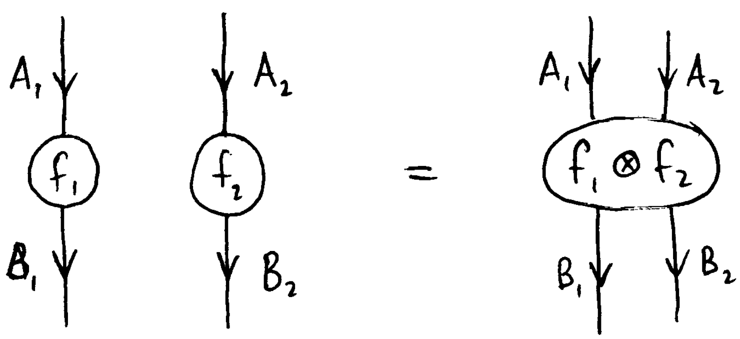
also drawn as



A <u>monoidal category</u> $\mathcal{T}$ has, first of all, a functor

$$\otimes : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{T}$$

We draw the tensor product of $f_1 : A_1 \to B_1$ & $f_2 : A_2 \to B_2$ as



We also want a "unit object" $I \in A$, associativity & left- and right unit laws ... <u>But</u> we don't want the most naive form of these. E.g. we <u>don't</u> want

$$(A \otimes B) \otimes C = A \otimes (B \otimes C)$$

We don't have this when $\otimes = \times$ in $\mathcal{T} = \mathrm{Set}$:

$$\begin{array}{ccc} (A \times B) \times C & \neq & A \times (B \times C) \\ \| & & \| \\ \{((a,b),c) : a \in A, b \in B, c \in C\} & & \{(a,(b,c)) : a \in A, b \in B, c \in C\} \end{array}$$

We have $(A \times B) \times C \cong A \times (B \times C)$, and better yet we have a specific isomorphism (a "natural" iso, in fact):

$$\alpha_{A,B,C}((a,b),c) = (a,(b,c))$$

So in general, a monoidal category $\mathcal{T}$ is equipped with a natural isomorphism

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$$

called the associator. This must satisfy the "pentagon identity":

$$\begin{tikzcd}
 & ((A \otimes B) \otimes C) \otimes D \arrow[dl, "\alpha_{A \otimes B, C, D}"'] \arrow[dr, "\alpha_{A,B,C} \otimes 1_D"] & \\
(A \otimes B) \otimes (C \otimes D) \arrow[ddr, "\alpha_{A,B,C \otimes D}"'] & & (A \otimes (B \otimes C)) \otimes D \arrow[d, "\alpha_{A, B \otimes C, D}"] \\
 & & A \otimes ((B \otimes C) \otimes D) \arrow[dl, "1_A \otimes \alpha_{B,C,D}"] \\
 & A \otimes (B \otimes (C \otimes D)) &
\end{tikzcd}$$

commutes.

The pentagon identity implies the same sort of thing for tensor products of >4 objects (Mac Lane's coherence theorem). Finally, we need natural isomorphisms

$$\ell_A : I \otimes A \longrightarrow A$$
$$r_A : A \otimes I \longrightarrow A$$

the left and right unitors, satisfying the "triangle identity":

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "r_A \otimes 1_B"'] & & A \otimes (I \otimes B) \arrow[dl, "1_A \otimes \ell_B"] \\
& A \otimes B &
\end{tikzcd}$$

commutes.

MacLane's coherence theorem says, given the pentagon identity and the triangle identity, <u>all</u> diagrams built up out of just α, r, and ℓ commute!

Examples of monoidal categories:

$(Set, \times, 1)$

$(Vect, \otimes, \mathbb{C})$

$(Vect, \oplus, \{0\})$

$(Hilb, \otimes, \mathbb{C})$

Homework: Suppose $\mathcal{C}$ is a category with finite products, i.e. given objects $A_1, \dots, A_n$ $(n \geq 0)$ there exists an object A with morphisms

$$p_i : A \longrightarrow A_i$$

such that for any $f_i : X \longrightarrow A_i$

$$\exists! \quad f : X \longrightarrow A$$

s.t.
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "f_i"'] & A \arrow[d, "p_i"] \\ & A_i \end{tikzcd}$$
commutes $\forall i$.

Show we can make $\mathcal{C}$ into a mon. cat. by choosing a product $A \times B$ for any pair of objects & defining
(w. $p_1 : A \times B \to A$, $p_2 : A \times B \to B$)

$A \otimes B$, choosing a terminal object (a product of <u>no</u> objects) 1 & defining $I = 1$.