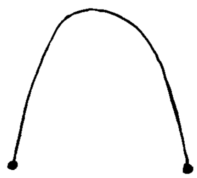
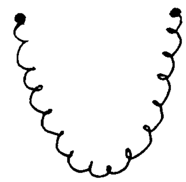
10 October 2006

Homework: "A spring in imaginary time!" at
http://math.ucr.edu/home/baez/classical/





There's a difference in sign which ultimately comes from $i^2 = -1$.

To do this homework, you'll use the Lagrangian approach to classical particle dynamics.

Suppose X is a manifold, the <u>configuration space</u>. We want a law of physics satisfied by paths

$$\gamma : [t_0, t_1] \longrightarrow X$$

-- the "Euler-Lagrange equation". To get this, we define

$$P_{x_0,x_1} X = \{ \gamma : [t_0,t_1] \longrightarrow X : \gamma(t_i) = x_i \}$$

which is an infinite dimensional manifold in its own right (in fact, a Frechet manifold), and we choose a smooth function

$$S : P_{x_0,x_1} X \longrightarrow \mathbb{R}$$

called the <u>action</u>.

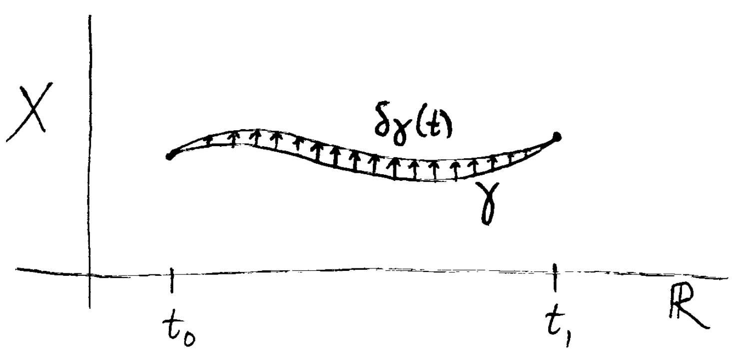
Abstractly, the Euler-Lagrange equation says

$$dS(\gamma) = 0$$

where $dS \in \Omega^1(P_{x_0,x_1}X)$. So, a particle will follow a path that's a critical point of the action. A bit more concretely, we have:

$$\forall \delta\gamma \qquad \underbrace{dS(\gamma)}_{\in\, T^*_\gamma(P_{x_0,x_1}X)}\underbrace{(\delta\gamma)}_{\in\, T_\gamma P_{x_0,x_1}X} = 0$$

where $\delta\gamma$ is a "variation" in γ, i.e. a tangent vector at $\gamma \in P_{x_0,x_1}X$.



Note: we can think of $\delta\gamma$ as a path in TX, with

$$\delta\gamma(t) \in T_{\gamma(t)}X \subseteq TX$$

In physics we often have actions of the form

$$S(\gamma) = \int_{t_0}^{t_1} L(\underbrace{\gamma(t)}_{\text{position}}, \underbrace{\dot{\gamma}(t)}_{\text{velocity}})\, dt$$

where the Lagrangian is some smooth function

$$L : \underbrace{TX}_{\text{position/velocity pairs}} \longrightarrow \mathbb{R}.$$

Let's see what $dS(\gamma) = 0$ says in this case. For all $\delta\gamma \in T_\gamma P_{x_0,x_1} X$, we have

$$\begin{aligned} 0 &= dS(\gamma)(\delta\gamma) \\ &= \int_{t_0}^{t_1} \frac{\partial L}{\partial x^i}\, \delta\gamma^i(t) + \frac{\partial L}{\partial y^i}\, \delta\dot{\gamma}^i(t)\, dt \end{aligned}$$

where x^i are local coordinates on X (which is locally $\cong \mathbb{R}^n$), which give local coordinates x^i, y^i on TX (locally $\cong T\mathbb{R}^n \cong \mathbb{R}^n \oplus \mathbb{R}^n$). Continuing this calculation:

$$\begin{aligned} 0 &= \int_{t_0}^{t_1} \frac{\partial L}{\partial x^i}\, \delta\gamma^i(t) + \frac{\partial L}{\partial y^i} \frac{d}{dt} \delta\gamma^i(t)\, dt \\ &\underset{\text{int. by parts}}{=} \int_{t_0}^{t_1} \frac{\partial L}{\partial x^i}\, \delta\gamma^i(t) - \left(\frac{d}{dt} \frac{\partial L}{\partial y^i}\right) \delta\gamma^i(t)\, dt \end{aligned}$$

The boundary terms in our integration by parts vanish since

$$\delta\gamma(t_0) = \delta\gamma(t_1) = 0$$

(see the picture). So:

$$0 = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial y^i} \right) \delta\gamma^i(t)\, dt$$

for all $\delta\gamma$. So we get

$$\frac{\partial L}{\partial x^i} = \frac{d}{dt} \frac{\partial L}{\partial y^i}$$

as the Euler-Lagrange equations. More pedantically:

$$\frac{\partial L}{\partial x^i}(\gamma(t), \dot\gamma(t)) = \frac{d}{dt} \frac{\partial L}{\partial y^i}(\gamma(t), \dot\gamma(t))$$

Physicists don't write (x^i, y^i) as coordinates on TX; they use $(q^i, \dot q^i)$ even though "$\dot q^i$" here is not the time derivative of anything. Also, they write $q : [t_0, t_1] \longrightarrow X$ instead of $\gamma : [t_0, t_1] \longrightarrow X$. Now $\dot q^i$ is ambiguous! But tough. The E-L equations look like:

$$\frac{\partial L}{\partial q^i} = \frac{d}{dt} \frac{\partial L}{\partial \dot q^i}$$

<u>Example</u>: a particle in a potential in $\mathbb{R}^n$. Here $X = \mathbb{R}^n$ and $L : TX \longrightarrow \mathbb{R}$ is given by:

$$\underbrace{L(\underset{\underset{X}{\cap}}{q}, \underset{\underset{T_qX}{\cap}}{\dot{q}})}_{} = \underbrace{\frac{m}{2}|\dot{q}|^2}_{\text{Kinetic energy}} - \underbrace{V(q)}_{\text{Potential energy}}$$

The Lagrangian is weird: it's not the sum of energies, but the difference:

$$L = \underbrace{\text{Kinetic}}_{\text{how much is happening}} - \underbrace{\text{Potential}}_{\text{how much could be happening that's not}} = \text{total "happeningness"}$$

Nature usually tries to minimize the integral of the Lagrangian (the "happeningness") over time. The E-L equations say:

$$\frac{\partial L}{\partial q^i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}$$

$$-\frac{\partial V}{\partial q^i} = \frac{d}{dt} m\dot{q}_i$$

$$-\frac{\partial V}{\partial q^i} = m\ddot{q}_i$$

or

$$F = ma$$

where $F_i = -\frac{\partial V}{\partial q^i}$ is the <u>force</u> & $a_i = \ddot{q}_i$ is the <u>acceleration</u>.

Quite generally, for any Lagrangian $L : TX \to \mathbb{R}$, we can define:

$$\frac{\partial L}{\partial q^i} = F_i \qquad \underline{\text{force}}$$

$$\frac{\partial L}{\partial \dot{q}^i} = p_i \qquad \underline{\text{momentum}}$$

so the E-L equations say

$$\frac{dp_i}{dt} = F_i \, .$$

To relate all this to cohomology, step back: we have derived classical mechanics from the principle of least action, based on

$$S : PX \longrightarrow \mathbb{R}$$

It would be great if there were a 1-form $\alpha \in \Omega^1(X)$ such that

$$S(\gamma) = \int_\gamma \alpha$$

Alas, the action S in our example is not of this form! Why not? Because in our example $S(\gamma)$ depends on the reparameterization of γ, for obvious physical reasons. But, we can write $S(\gamma)$ as the integral of some 1-form over a path in some other space. What's this other space? For any manifold M, T^*M has a God-given 1-form on it, called the canonical 1-form. We'll use this to get the job done, but with

$$M = \underbrace{X}_{\text{space}} \times \underbrace{\mathbb{R}}_{\text{time}}$$

— the extended configuration space.