Monoidal Categories
vs
Cartesian Categories
(categories with finite products)

Being cartesian is a property of a category (i.e. it is or it isn't). Being monoidal is not: it's an extra structure on a category. But, we can make a cartesian category into a monoidal one by choosing binary products and a terminal object.

What's so special about cartesian categories?

Cartesian monoidal	Noncartesian monoidal
$(\mathrm{Set}, \times, 1)$	$(\mathrm{Vect}, \otimes, \mathbb{C})$
$(\mathrm{Vect}, \oplus, 0)$	$(\mathrm{Hilb}, \otimes, \mathbb{C})$

In a sense, cartesian categories are more "classical" (or at least "intuitionistic") while noncartesian ones are more "quantum". In a cartesian category it's possible to duplicate data; in quantum mechanics it's not – you "can't clone a quantum" (Wooters-Zurek).

In set theory we duplicate data via

$$\begin{aligned} \Delta_S : S &\longrightarrow S \times S \\ s &\longmapsto (s,s) \end{aligned}$$

the diagonal map. In general, we can do this in any cartesian category:

$$\begin{tikzcd} A \arrow[rr, "\exists!"] \arrow[dr, "1_A"'] \arrow[drrr, "1_A"] & & A \times A \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ & A & & A \end{tikzcd}$$

We call the unique morphism making this diagram commute the diagonal, $\Delta_A : A \to A \times A$.

Note: "duplicating" via Δ_A and then "keeping one copy" via p_1 or p_2 is the same as doing nothing (1_A).

We can also "delete" information in a cartesian category, using

$$!_A : A \longrightarrow 1$$

the unique morphism to the terminal object.

The projections

$$p_1 : A \times B \to A$$

$$p_2 : A \times B \to B$$

delete information too; e.g. in Set

$$p_1 : (a,b) \longmapsto a$$

deletes information about b. But, these projections can be expressed in terms of $!_B$, $!_A$. For example,

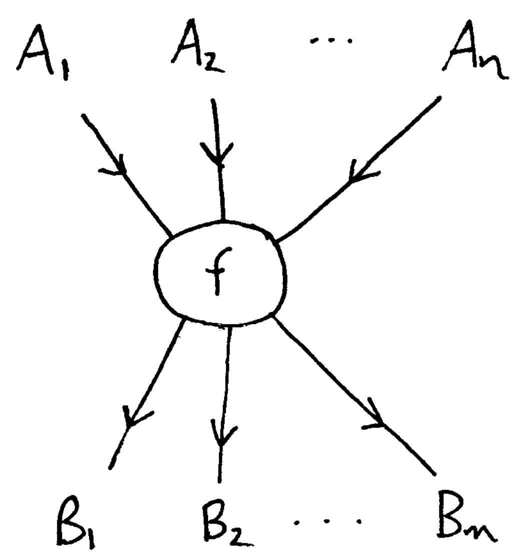
$$\begin{tikzcd} A \times B \arrow[r, "1_A \times !_B"] \arrow[rr, bend right, "p_1"'] & A \times 1 \arrow[r, "r_A", "\sim"'] & A \end{tikzcd}$$

commutes so the "deletion" in p_1 can be expressed in terms of $!_B$.

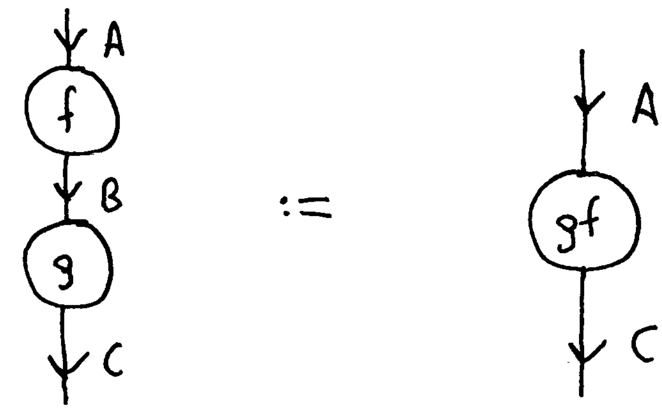
Recall: in any monoidal category we can draw morphisms

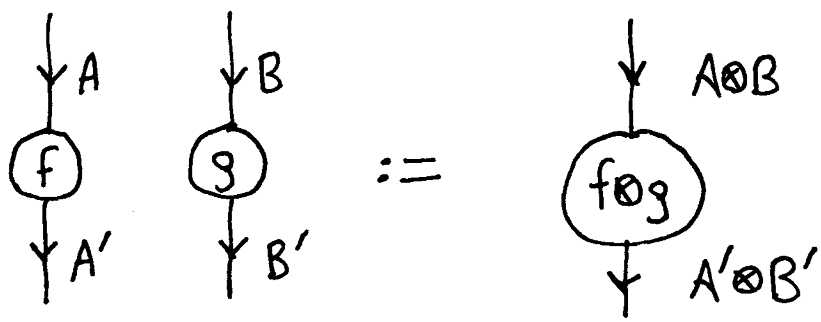
$$f: A_1 \otimes \cdots \otimes A_n \to B_1 \otimes \cdots \otimes B_m$$

as "string diagrams" like

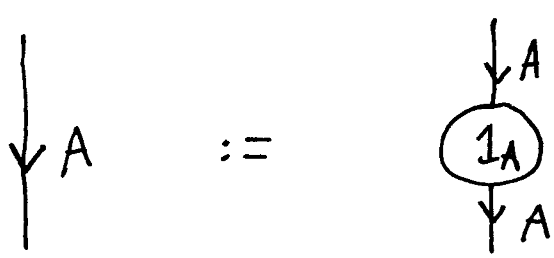


We draw composition & tensoring as:

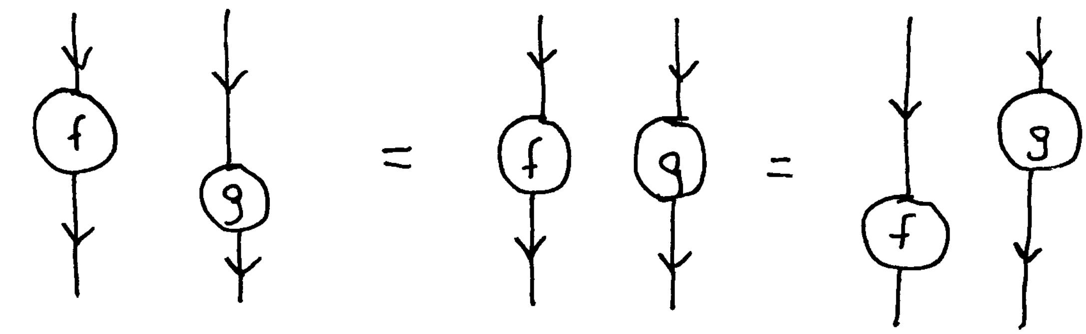




and identities as:



Exercise: why is



$$(1\otimes g)(f\otimes 1) = f\otimes g = (f\otimes 1)(1\otimes g) \ ?$$

This is called the <u>interchange law</u> – it's true in any monoidal category. It follows from: 1) $\otimes : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ is a functor, and 2) the definition of $\mathcal{C}\times\mathcal{C}$. In $\mathcal{C}\times\mathcal{C}$ we have

$$(f,g)(f',g') = (ff',gg')$$

so

$$(1,g)(f,1) = (f,g) = (f,1)(1,g)$$

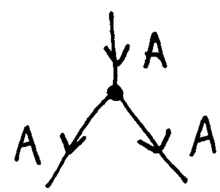
and then apply the functor $\otimes$ – voilà!

Now, how should we draw

$$\Delta_A : A \to A \times A$$

in the special case of a cartesian category?

Answer:



How about

$$!_A : A \to 1$$

Answer:

← this is a picture of A

← this is a picture of 1

They <u>look</u> like duplication & deletion!

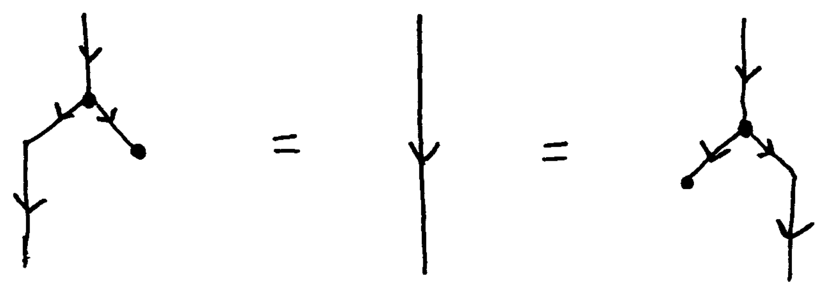
We saw

$$A \xrightarrow{\Delta_A} A \times A \xrightarrow{1_A \times !_A} A \times 1 \xrightarrow[\sim]{r_A} A$$

and

$$A \xrightarrow{\Delta_A} A \times A \xrightarrow{!_A \times 1_A} 1 \times A \xrightarrow[\sim]{\ell_A} A$$

are both the identity on A. In pictures:



Closed Monoidal Categories

Cartesian closed categories underlie the λ-calculus; more general monoidal closed categories underlie the "quantum λ-calculus", which is good (?) for quantum computation.

Set reigns over all categories since, given any category $\mathcal{C}$ and objects $A, B \in \mathcal{C}$, there is a <u>set</u> $\mathrm{Hom}(A,B)$ of all morphisms $f: A \to B$. But in many examples, like $\mathcal{C} =$ Vect, Top, Ab, ... we also have an <u>object in $\mathcal{C}$</u> $\mathrm{hom}(A,B)$ of morphisms $f: A \to B$. This is formalized by the notion of a "closed" category. We'll define a closed <u>monoidal</u> category.

Def. – A monoidal category $(\mathcal{C}, \otimes, I)$ is (left) closed if for any object $A \in \mathcal{C}$, the functor

$$\begin{aligned} A\otimes - : \mathcal{C} &\to \mathcal{C} \\ B &\mapsto A\otimes B \\ f &\mapsto 1_A \otimes f \quad (= A\otimes f) \end{aligned}$$

has a right adjoint, i.e. there's a natural isomorphism

$$\mathrm{Hom}(A\otimes X, Y) \cong \mathrm{Hom}(X, \mathrm{hom}(A,Y))$$

for some functor

$$\mathrm{hom}(A, -) : \mathcal{C} \to \mathcal{C}$$

called the internal hom.

Note: $\mathrm{Hom}(A,B) \in \mathrm{Set}$ while $\mathrm{hom}(A,B) \in \mathcal{C}$.

Example: $\mathcal{C} = \mathrm{Vect}$, $\otimes$ = usual tensor product.

Linear maps

$$f : A \otimes X \to Y$$

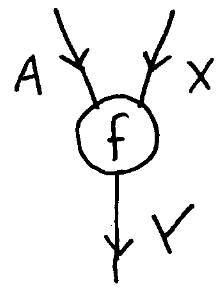
are in natural 1-1 correspondence with linear maps

$$\tilde{f} : X \to \hom(A, Y)$$

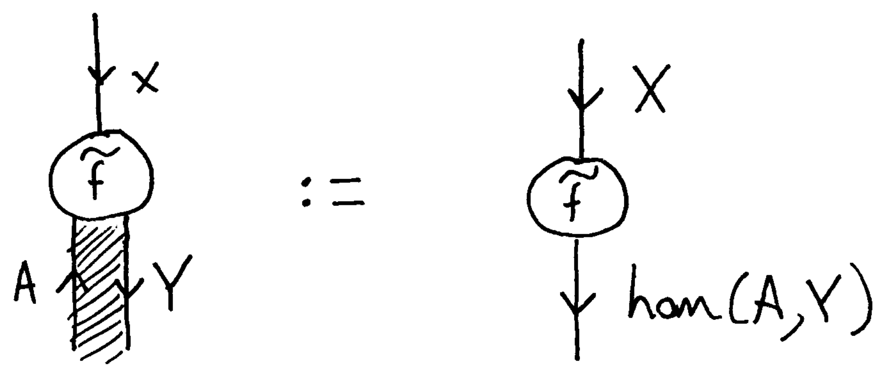
where $\hom(A, Y)$ is the <u>vector space</u> of linear maps from A to Y, and

$$\tilde{f}(a)(x) = f(a \otimes x).$$

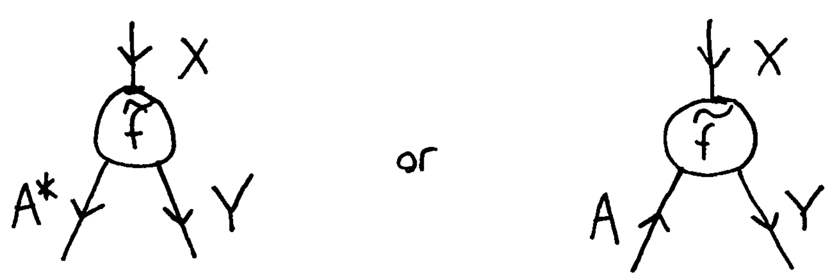
In our picture notation, we say morphisms



are in 1-1 correspondence with morphisms



Indeed, in Vect (or any <u>compact</u> closed monoidal category) we have $\hom(A,Y) \cong A^* \otimes Y$, so we can draw this picture as



where in physics the backwards arrow signifies an <u>antiparticle</u>!

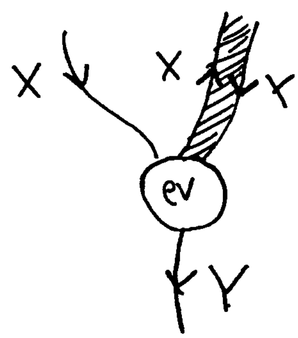
In any closed monoidal category the morphism

$$1 : \hom(X,Y) \to \hom(X,Y)$$

gives a morphism

$$ev : X \otimes \hom(X,Y) \to Y$$

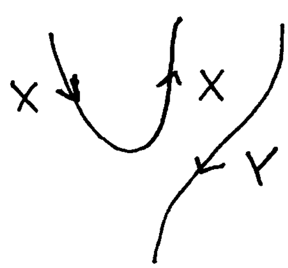
drawn as



In Vect we have

$$ev : x \otimes f \longmapsto f(x)$$

or



Particle/antiparticle annihilation!