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From Lagrangian to Hamiltonian Dynamics

We want a description of classical mechanics where the action is the integral of some 1-form along a path. This path will lie not

$$X = \text{config. space} \quad \ni \text{position}$$

nor in

$$TX \qquad \ni (\text{position}, \text{velocity})$$

nor in

$$T^*X = \text{phase space} \quad \ni (\text{position}, \text{momentum})$$

but in

$$T^*(X \times \mathbb{R}) = \text{extended phase space} \quad \ni (\text{position}, \text{momentum}, \text{time}, \text{energy})$$

To get there, let's first study the phase space T^*X and energy (the Hamiltonian). We start with a Lagrangian

$$L : TX \longrightarrow \mathbb{R}$$

& get Euler-Lagrange equations:

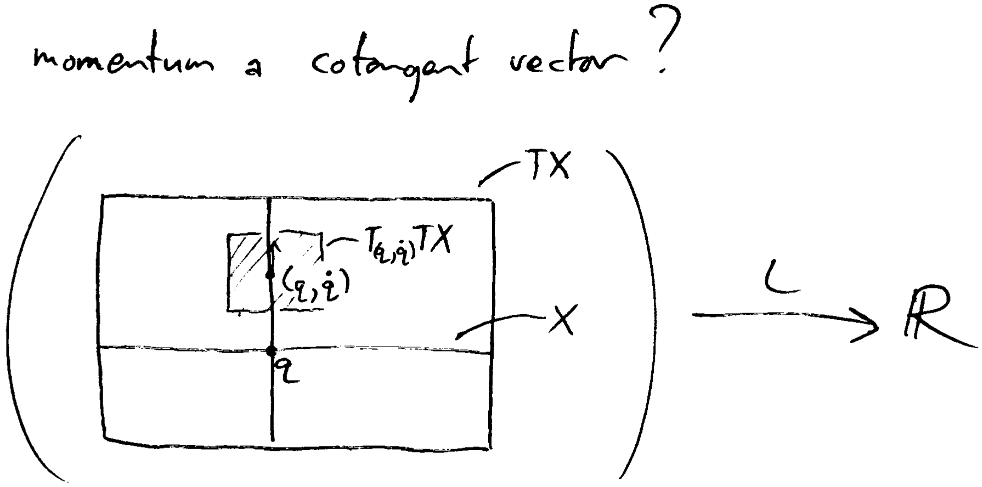
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i}$$

where

$$\frac{\partial L}{\partial \dot{q}^i} = p_i = \text{"momentum"}$$

$$\frac{\partial L}{\partial q^i} = F_i = \text{"force"}.$$

Why is momentum a cotangent vector?



$\frac{\partial L}{\partial \dot q_i}$ describes derivative of L in the "vertical" direction.

$$\{\text{Vertical vectors}\} \subseteq T_{(q,\dot q)}TX$$

where

$$\{\text{Vertical vectors}\} = \ker d\pi$$
$$\cong$$
$$T_{(q,\dot q)}T_q X$$

where $d\pi : T_{(q,\dot q)}TX \longrightarrow T_q X$ is the differential of the projection $\pi : TX \longrightarrow X$.

$$(q,\dot q) \longmapsto q$$

But note for any vector space V, $T_v V \cong V$ in a canonical way so $T_{(q,\dot q)}T_q X \cong T_q X$. So momentum really is the derivative of $L : TX \longrightarrow \mathbb{R}$, but only in vertical directions. The derivative of L is the 1-form dL:

$$dL(q,\dot q) : T_{(q,\dot q)}TX \longrightarrow \mathbb{R}$$

so the momentum is this restricted to vertical vectors:

$$\begin{array}{c}\{\text{Vertical vectors}\} \subseteq T_{(q,\dot q)}TX \\ \text{\rotatebox{90}{$\cong$}} \\ T_q X\end{array}$$

So the momentum is a linear map

$$p : T_q X \longrightarrow \mathbb{R}$$

i.e. $p \in T_q^* X$ is a <u>cotangent vector</u>.

Now we'll switch from the Lagrangian approach, based on

$$(q, \dot q) \in TX$$

to the "Hamiltonian" approach based on

$$(q, p) \in T^* X = \text{"phase space"}$$

We'll do this using the <u>Legendre transform</u>

$$\begin{aligned} \lambda : TM &\longrightarrow T^*M \\ (q, \dot q) &\longmapsto (q, p) \end{aligned}$$

where

$$p_i = \frac{\partial L}{\partial \dot q^i}$$

(Note λ is defined using L). From now on, let's assume L is <u>strongly regular</u>, meaning that

$$\lambda : TM \longrightarrow T^*M$$

is a diffeomorphism.

Example: A particle moving on a Riemannian manifold (X, g) has Lagrangian

$$L(q,\dot q) = \frac{m}{2} g(\dot q, \dot q) - V(q)$$

where $m > 0$ & $V : X \to \mathbb{R}$. Here

$$p_i = \frac{\partial L}{\partial \dot q_i} = m g_{ij} \dot q^j$$

where $g_{ij}\dot q^j$ is the "coordinate-ridden" expression for

$$g(\dot q, -) \in T_q^* X$$

So λ is a diffeo., since g is nondegenerate: i.e.

$$\dot q \longmapsto g(\dot q, -)$$

is 1-1 and onto. So this L is strongly regular.

To translate the Euler-Lagrange equations into eqns satisfied by q & p, we need the concept of energy.

<u>Thm</u> (conservation of energy) Given any $L : TX \to \mathbb{R}$ and $q : [t_0, t_1] \to X$ satisfying the E-L eqns, then

$$E(q(t), \dot q(t)) \qquad (\dot q(t) = \tfrac{d}{dt} q(t))$$

is independent of t, where $E : TX \to \mathbb{R}$ is given by

$$\begin{aligned} E &= p_i \dot q^i - L(q, \dot q) \\ &= \frac{\partial L}{\partial \dot q^i} \dot q^i - L(q, \dot q) \end{aligned}$$

Proof: $\frac{d}{dt} E(q(t),\dot q(t)) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q^i}\right)\dot q^i + \frac{\partial L}{\partial \dot q^i}\frac{d}{dt}\dot q^i - \frac{\partial L}{\partial q^i}\frac{dq^i}{dt} - \frac{\partial L}{\partial \dot q^i}\frac{d}{dt}\dot q^i = 0,$

since the E-L eqns. say $\frac{d}{dt}\frac{\partial L}{\partial \dot q^i} = \frac{\partial L}{\partial q^i}$. ■

Example: For a particle on X with

$$L = \frac{m}{2} g(\dot q,\dot q) - V(q)$$

we have

$$\begin{aligned} E &= p_i \dot q^i - L \\ &= m g_{ij}\dot q^i \dot q^j - \frac{m}{2} m g_{ij}\dot q^j \dot q^i + V(q) \\ &= \frac{m}{2} g(\dot q,\dot q) + V(q) \end{aligned}$$

So, 'energy' is kinetic energy plus potential energy, just as L was the difference.

Using our diffeomorphism

$$\lambda : TX \xrightarrow{\sim} T^*X$$

we can define the Hamiltonian $H := E \circ \lambda^{-1}$. I.e.

$$H(q,p) = E(q,\dot q)$$

where $\lambda(q,\dot q) = (q,p)$.

Now let's figure out "Hamilton's eqns" describing the time evolution of (q,p), given that $(q,\dot q)$ satisfy the E-L eqns.

$$\begin{gathered} \| \\ (q(t),p(t)) \\ \| \\ \lambda\left(q(t),\frac{dq}{dt}(t)\right) \end{gathered}$$

Here's how...

Compute dH in two ways

1) $H : T^*X \to \mathbb{R}$ and T^*X has local coordinates (q^i, p_i) coming from the local coordinates q^i on X. So we get

$$dH = \frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial p_i} dp_i$$

2) We can also compute dH using coordinates $(q^i, \dot{q}^i)$. These are local coords. on TX coming from q^i on X, but they may be viewed as local coords. on T^*X using

$$\lambda : TX \xrightarrow{\sim} T^*X.$$

Here we get

$$\begin{aligned} dH &= d(p_i \dot{q}^i - L(q, \dot{q})) \\ &= dp_i \dot{q}^i + p_i d\dot{q}^i - \frac{\partial L}{\partial q^i} dq^i - \frac{\partial L}{\partial \dot{q}^i} d\dot{q}^i \\ &= dp_i \dot{q}^i - \frac{\partial L}{\partial q^i} dq^i \end{aligned}$$

where we recall $p_i = \frac{\partial L}{\partial \dot{q}^i}$. Comparing coefficients thus gives

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \qquad \frac{\partial L}{\partial q^i} = -\frac{\partial H}{\partial q^i}$$

or, using $\frac{\partial L}{\partial q^i} = \frac{d}{dt} p_i$,

$$\boxed{\dot{q}^i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}}$$

HAMILTON'S EQUATIONS