24 October 2006

Hamiltonian Mechanics & Symplectic Geometry

We've seen that 'any' Lagrangian $L: TX \to \mathbb{R}$ gives Euler-Lagrange equations describing the flow on TX — "time evolution":

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$

We also get a Legendre transform

$$\begin{aligned} \lambda : TM &\longrightarrow T^*M \\ (q,\dot{q}) &\longmapsto (q,p) \end{aligned}$$

where $p_i = \frac{\partial L}{\partial \dot{q}^i}$. Assume L is strongly regular, i.e. λ is a diffeomorphism. Then we get a flow on T^*M describing time evolution of position and momentum, which satisfies Hamilton's equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}$$

where $H: T^*M \to \mathbb{R}$ is the Hamiltonian given by

$$H(q,p) = p_i \dot{q}^i - L(q,\dot{q})$$

where $(q,\dot{q}) = \lambda^{-1}(q,p)$.

Example: If (X, g) is a Riemannian manifold and $V : X \to \mathbb{R}$ is the potential energy, then the Hamiltonian for a particle of mass m on X is

$$H(q,p) = \frac{|p|^2}{2m} + V(q)$$

Here Hamilton's equations say, first:

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \text{or} \quad \dot{q}^i = \frac{p^i}{m}$$

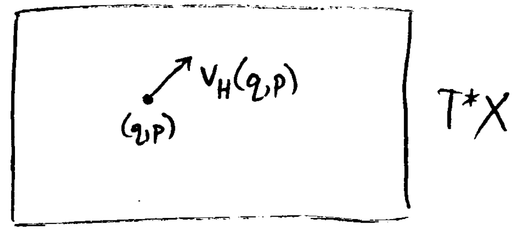
where $p^i = g^{ij} p_j$ (i.e. the metric $g : T_qX \times T_qX \to \mathbb{R}$ gives $\flat : T_qX \xrightarrow{\sim} T_q^*X$ by $v \mapsto g(v, -)$, and we use this to turn the cotangent vector p into the tangent vector $\flat^{-1}(p) =: \#(p)$, with components p^i.)
This equation $p^i = m\dot{q}^i$ is boring; the interesting equation is Hamilton's 2nd eqn:

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}, \quad \text{or} \quad \dot{p}_i = -\frac{\partial V}{\partial q^i}.$$

I.e. $F = ma$.

Let's seek a coordinate-free formulation of Hamilton's equations. Hamilton's eqns give a vector field v_H on T^*X describing how $(q(t), p(t))$ moves under the Hamiltonian flow. This vector field v_H is called the Hamiltonian vector field.

$$\frac{d}{dt}(q(t),p(t)) = \underset{\substack{\cap \\ T(T^*X)}}{v_H(q(t),p(t))}$$

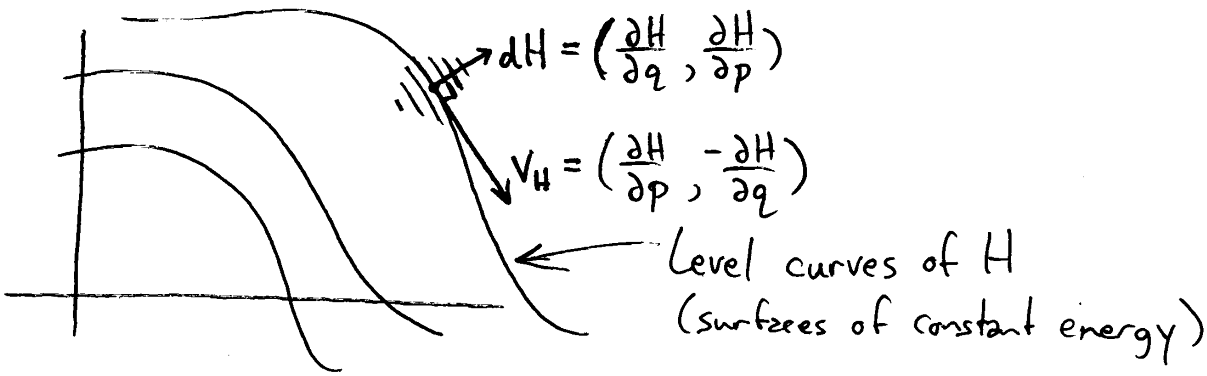


where

$$v_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} + \frac{-\partial H}{\partial q^i}\frac{\partial}{\partial p_i}$$

where $\{\frac{\partial}{\partial q^i}, \frac{\partial}{\partial p_i}\}$ is a local coordinate basis of vector fields on T^*X.

Example: If $X = \mathbb{R}$ $T^*X = \mathbb{R}^2$



Here dH is really a 1-form, but in $\mathbb{R}^2$ we can use the metric to turn it into a vector $\vec{\nabla} H$ and rotate this 90° to get v_H. So any solution of Hamilton's equations moves along the level curves of H: this is conservation of energy.

What's the coordinate-free description of how to turn the 1-form dH into the vector field v_H? One way to turn 1-forms into vector fields is to use a (Riemannian) metric:

$$\begin{aligned} \flat : TM &\longrightarrow T^*M \\ v &\longmapsto g(v,-) \end{aligned}$$

has an inverse when g is nondegenerate, so we get

$$\# : T^*M \longrightarrow TM$$

But, $\# dH = \vec{\nabla} H$ is perpendicular to the level curves of H, not tangent. Instead of a metric, let's use an antisymmetric nondegenerate bilinear form

$$\varpi : TM \times TM \longrightarrow \mathbb{R}$$

(where in our example $M = T^*X$). In our example

$$\varpi = dq_i \wedge dp^i$$

[Homework: Show that, with this 2-form on T^*X we get

$$\varpi(v_H, -) = dH$$

I.e. now we define

$$\begin{aligned} \flat : TM &\longrightarrow T^*M \\ v &\longmapsto \varpi(v,-) \end{aligned}$$

and if $\flat$ is an isomorphism we say ϖ is nondegenerate and we set $v_H = \#(dH)$.]

In fact, the 2-form

$$\varpi = dq^i \wedge dp_i$$

on T^*X can be defined without using coordinates or any metric on X. How? Notice that

$$\varpi = -d\alpha$$

where

$$\alpha = p_i dq^i .$$

We'll see in fact that this α is a 1-form on T^*X — the canonical 1-form on T^*X — which can be defined without coordinates. How do we define α without coordinates? α likes to eat tangent vectors to T^*X, e.g.

$$v \in T_{(q,p)} T^*X .$$

The projection

$$\begin{aligned} \pi : T^*X &\longrightarrow X \\ (q,p) &\longmapsto q \end{aligned}$$

gives

$$d\pi : T(T^*X) \longrightarrow TX$$

So, we can form

$$d\pi(v) \in T_q X$$

but $p \in T_q^*X$, so we get $p(d\pi(v)) \in \mathbb{R}$.

So: define

$$\alpha(v) = p(d\pi(v)) \quad \text{for } v \in T_{(q,p)}T^*X.$$

Next time, we'll see this agrees with

$$\alpha = p_i dq^i.$$