26 October 2006

Currying & Uncurrying, Evaluation & Coevaluation

Last time I talked about <u>left</u> closed monoidal categories, where

$$\mathrm{Hom}(A \otimes X, Y) \underset{\substack{\uparrow \\ \text{(natural)}}}{\cong} \mathrm{Hom}(X, \mathrm{hom}(A, Y))$$

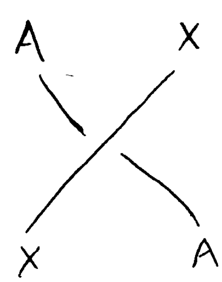
where $\mathrm{Hom}(-,-)$ is the set of morphisms & $\mathrm{hom}(-,-)$, the internal hom, is an object in the category. But actually, I like <u>right</u> closed monoidal categories, where

$$\mathrm{Hom}(X \otimes A, Y) \underset{\text{(nat.)}}{\cong} \mathrm{Hom}(X, \mathrm{hom}(A, Y)).$$

For most of our favorite examples, e.g. any cartesian category, or $(\mathrm{Vect}, \otimes)$ or $(\mathrm{Hilb}, \otimes)$, we have

$$A \otimes X \underset{\text{(nat.)}}{\cong} X \otimes A$$

— so left closed $\Longleftrightarrow$ right closed. These examples are all "braided" monoidal categories (as defined in Some Definitions Everyone Should Know) where we have an iso



$$B_{A,X} : A \otimes X \longrightarrow X \otimes A$$

we get

$$\begin{tikzcd}
f \arrow[d, bend right, "\text{``currying''}"'] & X \otimes A \longrightarrow Y & \underset{\sim}{g} \\
\tilde{f} & X \longrightarrow \hom(A,Y) & g \arrow[u, bend right, "\text{``uncurrying''}"']
\end{tikzcd}$$

i.e. a natural 1-1 correspondence, called currying after Haskell Curry.

In Set, currying turns this:

$$(x,a) \longmapsto f(x,a)$$ ⟵ a way of naming $f : X \times A \longrightarrow Y$

into this:

$$x \longmapsto (a \longmapsto f(x,a))$$ ⟵ a way of naming $\tilde{f} : X \longrightarrow \hom(A,Y)$

In any right closed monoidal category we could say currying turns:

$$x \otimes a \longmapsto f(x \otimes a)$$ ⟵ a way of naming $f : X \otimes A \longrightarrow Y$

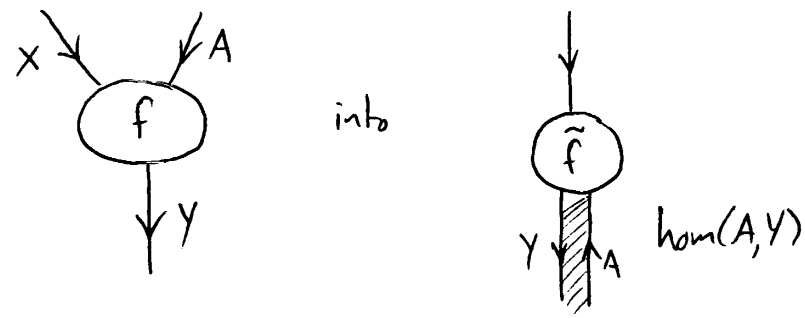
into

$$x \longmapsto (a \longmapsto f(x \otimes a))$$ ⟵ a way of naming $\tilde{f} : X \longrightarrow \hom(A,Y)$

Note:

$$\underset{\sim}{(\tilde{f})} = f \qquad \& \qquad \widetilde{(\underset{\sim}{g})}$$

In diagrams, currying turns



Suppose $X = I$, the unit for the tensor product:

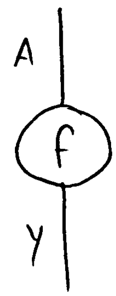
$$\begin{array}{ll} f & A \longrightarrow Y \\ \downarrow & \overline{} \\ f \circ \ell_A & I \otimes A \longrightarrow Y \\ \downarrow & \overline{} \\ \widetilde{f \circ \ell_A} & I \longrightarrow hom(A,Y) \end{array}$$

In Set, a morphism $g: I \longrightarrow Z$ is just an element of the set Z, so ~~in~~ general (especially the cartesian case) let's call a morphism $g: I \to Z$ an <u>element</u> of Z. (Exercise: see what this gives for $(Vect, \otimes)$ & $(Vect, \oplus)$).

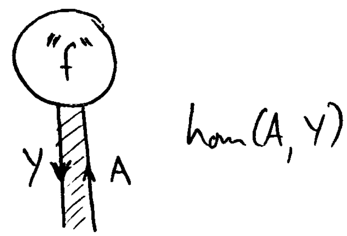
So from any morphism $f: A \to Y$ we get an element of $hom(A,Y)$, namely $\widetilde{f \circ \ell_A}$, or the <u>name</u> of f, denoted "f":

$$\text{“}f\text{”} : I \longrightarrow hom(A,Y)$$

In diagrams, from



we get the "name" of f:



<u>Another example</u>: $X = \hom(A, Y)$.

$$\frac{\hom(A,Y)\otimes A \longrightarrow Y}{\hom(A,Y) \longrightarrow \hom(A,Y)} \qquad \begin{matrix} ev \\ \big\uparrow \ \text{uncurrying} \\ 1 \end{matrix}$$

uncurrying the identity gives $ev : \hom(A,Y)\otimes A \longrightarrow Y$, called <u>evaluation</u> (or <u>application</u>).

In **Set**, ev is given by

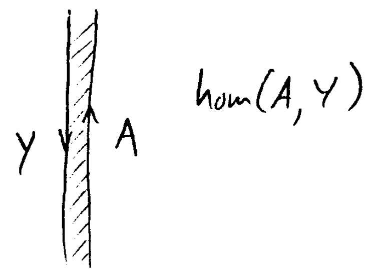
$$(f, a) \longmapsto f(a)$$

To check this, curry it!

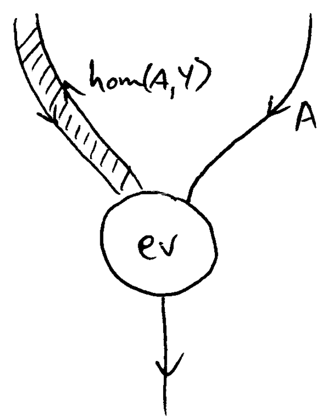
$$f \longmapsto \underbrace{(a \longmapsto f(a))}_{= f}$$

(η-reduction in λ-calculus jargon)

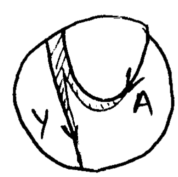
In terms of diagrams:



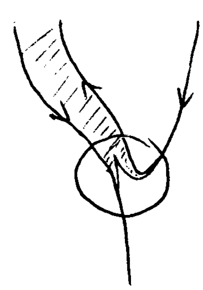
uncurries to



We can draw ev. as:

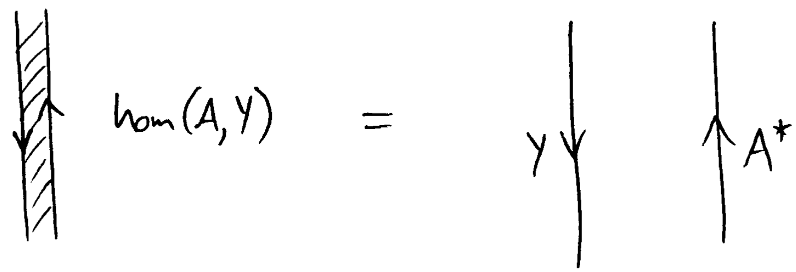


This is just a cute mnemonic:

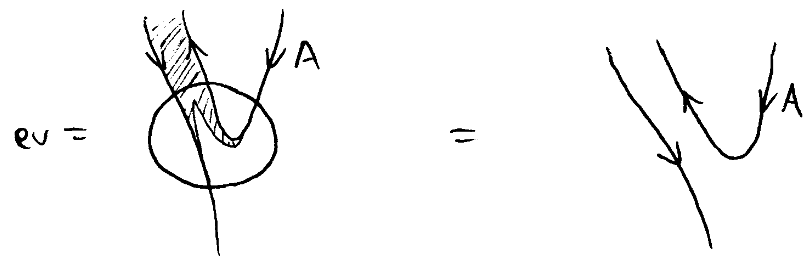


unless our category is <u>compact</u>, meaning every object A has a dual A^* s.t. $\hom(A,Y) \cong Y \otimes A^* \quad \forall Y$.

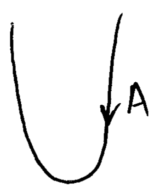
i.e.



& then



where



is the <u>counit</u> $e : A^* \otimes A \longrightarrow I$, which any compact category has.

In Vect, the counit is

$$\begin{aligned} e : A^* \otimes A &\longrightarrow \mathbb{C} \\ \ell \otimes v &\longmapsto \ell(v) \end{aligned}$$

(Check that in Vect $ev : hom(A,Y) \otimes A \longrightarrow \mathbb{C}$ really is given by

$$hom(A,Y) \otimes A \simeq Y \otimes A^* \otimes A \xrightarrow{1_Y \otimes e} Y \otimes \mathbb{C} \xrightarrow{r_Y} Y \text{)}$$

<u>Moral</u>: the diagrams can be taken more literally in quantum mechanics (Hilb) than in classical (Set).

Now let $Y = X \otimes A$:

$$\mathrm{curry}\ \Big\downarrow \ \frac{X \otimes A \longrightarrow X \otimes A}{X \longrightarrow \hom(A, X \otimes A)} \qquad 1 \mapsto \mathrm{coev}$$

where "coevaluation"

$$\mathrm{coev}: X \longrightarrow \hom(A, X \otimes A)$$

in Set is just

$$x \longmapsto (a \longmapsto (x, a))$$

To check this, uncurry it! We get the identity

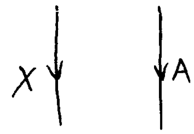
$$(x, a) \longmapsto (x, a).$$

In any (right) closed monoidal category, coev can be written:

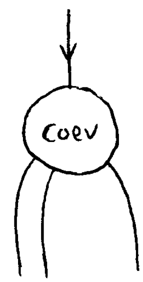
$$x \longmapsto (a \longmapsto x \otimes a)$$

This is an example of a term in the nascent "quantum λ-calculus" — the fragment of the λ-calculus that makes sense in any closed monoidal category.

In diagrams, we get coev by currying

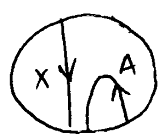


by "pulling down the right-hand wire"

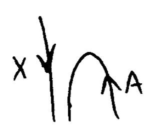


$$\text{coev}: X \longrightarrow \hom(A, X \otimes A)$$

We can draw coev as



& again we "pop the bubble" and draw this as



in a compact category like Vect, where we have

the <u>unit</u> $i : I \longrightarrow A \otimes A^* = \hom(A,A)$

In Vect, this is

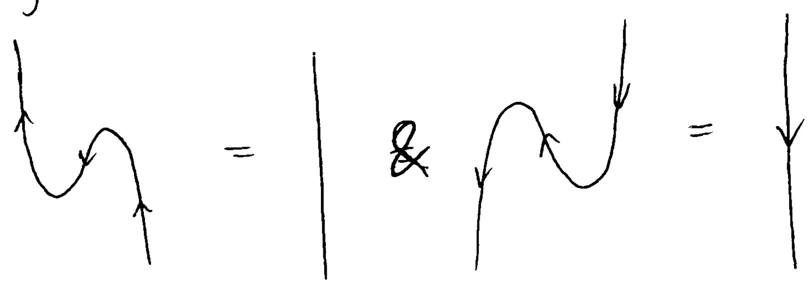
$$\begin{aligned} i : \mathbb{C} &\longrightarrow A \otimes A^* \\ 1 &\longmapsto \text{"}1_A\text{"} \end{aligned}$$

(Indeed, in any compact category we have

$$e : A^* \otimes A \longrightarrow I$$ 

$$i : I \longrightarrow A \otimes A^*$$ 

which satisfy two identities:

)