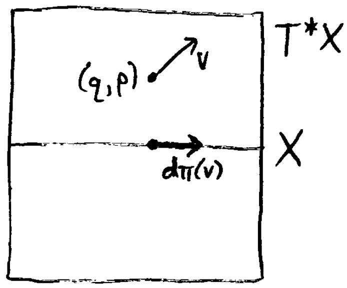
31 October 2006

The canonical 1-form on T^*X

We saw that for any manifold X, there's a God-given 1-form $\alpha \in \Omega^1(T^*X)$. With this setup:



$$q \in X$$
$$p \in T_q^*X$$
$$(q,p) \in T^*X$$
$$v \in T_{(q,p)}T^*X$$
$$d\pi : TT^*X \longrightarrow TX$$

α is given by:

$$\alpha(v) = p(d\pi(v)).$$

<u>Thm:</u> If q^i are any local coordinates on X, & (q^i, p_i) the corresponding coordinates on T^*X, then locally

$$\alpha = p_i dq^i$$

Pf: Choose any $v \in T_{(q,p)}T^*X$. Then

$$v = a^i \frac{\partial}{\partial q^i} + b_i \frac{\partial}{\partial p_i}$$

$\exists\, a^i, b_i \in \mathbb{R}$. We have:

$$p_i dq^i(v) = p_i dq^i\left(a^j \frac{\partial}{\partial q^j} + b_j \frac{\partial}{\partial p_j}\right) = a^i p_i$$

since $dq^i\left(\frac{\partial}{\partial q^j}\right) = \delta^i_j$ and $dq^i\left(\frac{\partial}{\partial p_j}\right) = 0$. On the other hand:

$$\begin{aligned}
\alpha(v) &= p\left(d\pi\left(a^j \frac{\partial}{\partial q^j} + b_j \frac{\partial}{\partial p_j}\right)\right) \\
&= p\left(a^j \frac{\partial}{\partial q^j}\right) \qquad\qquad p = p_i dq^i \\
&= a^j p_j
\end{aligned}$$

which is the same. ▨

There are various related ways α shows up in classical mechanics. We've seen

$$d\alpha = dp_i \wedge dq^i = -\varpi$$

where ϖ is a symplectic structure on T^*X:

Def: A <u>symplectic structure</u> on a manifold M is a 2-form ϖ on M s.t.:

1) ϖ is closed: $d\varpi = 0$
2) ϖ is <u>nondegenerate</u>: the map

$$\begin{aligned}
\flat : T_x M &\longrightarrow T^*_x M \\
v &\longmapsto \varpi(v, -)
\end{aligned}$$

is 1-1 (and thus also onto): $\forall v \neq 0\ \exists u\ \varpi(v,u) \neq 0$.

[<u>Homework</u>: Show that if α is the canonical 1-form on T^*X and $\omega = -d\alpha$, then ω is nondegenerate.]

In fact, a symplectic structure on M is enough to do classical mechanics w. M as "phase space" — the space of states of our system. Why? Roughly, the nondegeneracy of ω lets us define a <u>Hamiltonian vector field</u> v_H on M from any fn. $H: M \to \mathbb{R}$ (the Hamiltonian): let

$$\#: T_x^*M \longrightarrow T_xM$$

be the inverse of $\flat$ and

$$v_H = \#(dH)$$

This lets us write down <u>Hamilton's equations</u> describing the time evolution of states: given $x(t) \in M$ ($t \in [t_0, t_1]$) we say it satisfies Hamilton's equations if

$$\boxed{\frac{dx(t)}{dt} = v_H(x(t))}.$$

What does ω being <u>closed</u> do for us? Under mild assumptions on H we get a <u>flow</u> on M.

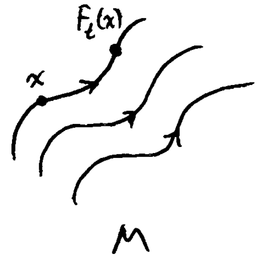
A flow on M is

$$F : \mathbb{R} \times M \longrightarrow M$$
$$(t, x) \longmapsto F_t(x)$$

s.t.

$$F_t(F_s(x)) = F_{t+s}(x)$$

& $F_0(x) = x$



How do we get this flow? The flow describes time evolution and satisfies Hamilton's equations

$$\frac{d}{dt} F_t(x) = v_H(F_t(x)).$$

The condition $d\omega = 0$ says that for any H, $F_t : M \to M$ preserves the symplectic structure ω!

$$F_t^* \omega = \omega$$

But there's another related way that $\alpha \in \Omega^1(T^*X)$ shows up in physics. We can use it to describe the action of a path

$$q : [t_0, t_1] \longrightarrow X$$

configuration space

How? We get

$$(q, \dot{q}) : [t_0, t_1] \longrightarrow TX$$
$$t \longmapsto (q(t), \dot{q}(t))$$

and thus

$$\gamma : [t_0, t_1] \longrightarrow T^*X \quad \text{~ phase space}$$
$$t \longmapsto \lambda(q(t), \dot{q}(t))$$

if we choose a Lagrangian $L : TX \to \mathbb{R}$ & use it to define the Legendre transform $\lambda : TX \to T^*X$. The action of our path q is

$$S(q) = \int_{t_0}^{t_1} L(q(t), \dot{q}(t))\, dt$$

$H = p_i \dot{q}^i - L$

$$= \int_{t_0}^{t_1} \Big(p_i(t)\dot{q}^i(t) - H(q(t), p(t)) \Big) dt$$

$$= \int_{t_0}^{t_1} \Big(p_i(t) \frac{dq^i(t)}{dt} - H(q(t), p(t)) \Big) dt$$

$$= \int_{t_0}^{t_1} p_i(t) dq^i(t) - H(q(t), p(t))\, dt$$

$$= \int_\gamma \alpha - \int_{t_0}^{t_1} H(q(t), p(t))\, dt$$

So, we see that the principle of least action:

$$\delta S(\gamma) = 0$$

is equivalent to the principle

$$\delta \int_\gamma \alpha = 0$$

<u>if</u> we let γ vary only over path for which energy is <u>conserved</u>:

$$\{ \gamma : [t_0, t_1] \longrightarrow T^*X \quad : \quad q(t_i) = q_i \ \ \& \ \ H(q(t), p(t)) = E \}$$

for some $q_0, q_1 \in M$ & some energy $E \in \mathbb{R}$.