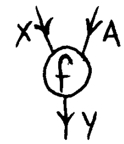
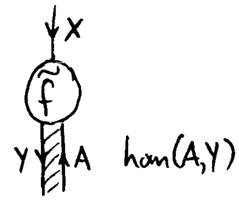
Let $\mathcal{C}$ be a monoidal closed category. Given

$$f : X \otimes A \longrightarrow Y$$



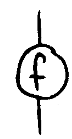
we can curry it & get

$$\tilde{f} : X \longrightarrow \hom(A, Y)$$



In particular, given

$$f : A \longrightarrow Y$$

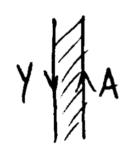


we get its <u>name</u>

$$\text{“}f\text{”} : 1 \longrightarrow \hom(A, Y)$$

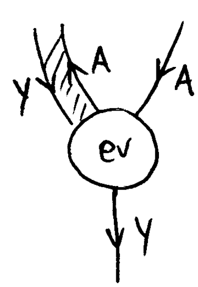
Also, we can uncurry, e.g. uncurrying

$$1 : \hom(A, Y) \longrightarrow \hom(A, Y)$$

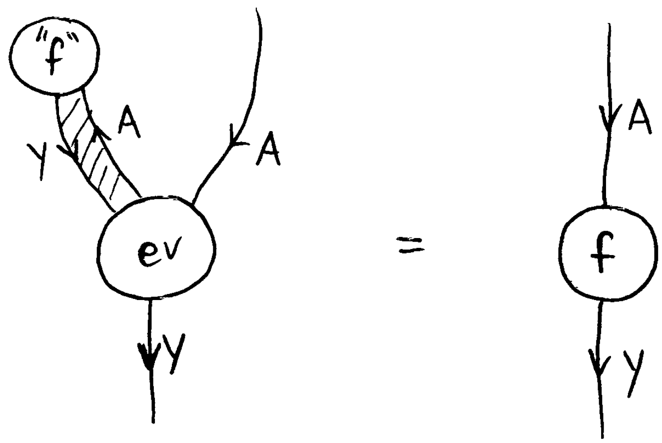


we get "evaluation"

$$ev : \hom(A, Y) \otimes A \longrightarrow Y$$



Theorem:



(translation:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\ell_A^{-1}"'] & Y \\ I \otimes A \arrow[r, "\text{"}f\text{"} \otimes 1_A"'] & \hom(A,Y) \otimes A \arrow[u, "ev"'] \end{tikzcd}$$

commutes.)

Proof: We use the naturality of currying

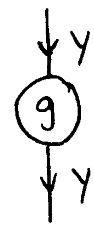
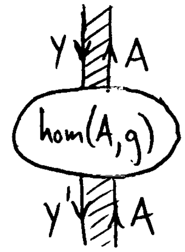
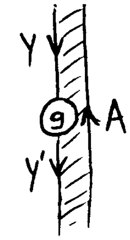
$$\mathrm{Hom}(X \otimes A, Y) \xrightarrow{\sim} \mathrm{Hom}(X, \hom(A,Y))$$

What does this mean? First, $- \otimes A$ & $\hom(A,-)$ are functors from $\mathcal{C}$ to $\mathcal{C}$. For $\hom(A,-)$, given $g: Y \to Y'$ we get

$$\hom(A,g): \hom(A,Y) \to \hom(A,Y')$$

e.g. for $\mathcal{C} = \mathrm{Set}$ or Vect,

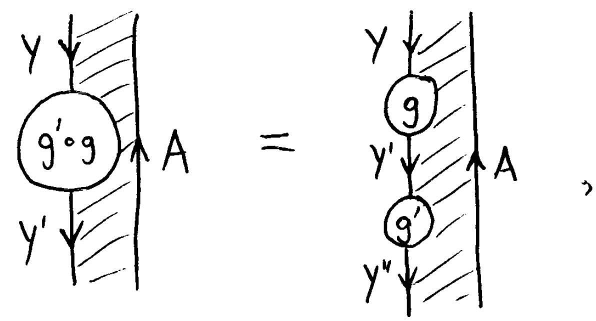
$$\hom(A,g): f \mapsto g \circ f.$$

In diagrams, given  we write  as  .

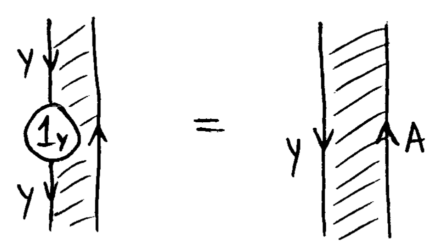
Then the fact that $\hom(A,-)$ is a functor says given

$$y \xrightarrow{g} y' \xrightarrow{g} y''$$

we have



and similarly identities are preserved:



So, $\hom(A,-)$ is a functor. Also:

$$\mathrm{Hom} : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathrm{Set}$$

is a functor, where $\mathcal{C}^{op}$ is the category with objects of $\mathcal{C}$ as objects, but all arrows reversed. Why?

Given a morphism

$$(g, g') : (A, A') \longrightarrow (B, B') \quad \text{in } \mathcal{C}^{op} \times \mathcal{C}$$

we get a function

$$\mathrm{Hom}(g,g') : \mathrm{Hom}(A,A') \longrightarrow \mathrm{Hom}(B,B')$$

In $\mathcal{C}$ we have $g' : A' \to B'$ but because of the "op", $g : B \to A$. For example, if $\mathcal{C} = \mathrm{Set}$ or Vect:

$$\begin{aligned} \mathrm{Hom}(g,g') : \mathrm{Hom}(A,A') &\longrightarrow \mathrm{Hom}(B,B') \\ f &\longmapsto g' \circ f \circ g \end{aligned}$$

(Check: Hom is a functor)

So currying is a natural isomorphism

$$\mathrm{Hom}(- \otimes A, --) \longrightarrow \mathrm{Hom}(-, \mathrm{hom}(A,--))$$

where both sides are functors $\mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Set}$. Naturality in $-$ means: given

$$g : X' \to X$$

this square commutes:

$$\begin{tikzcd} \mathrm{Hom}(X \otimes A, Y) \arrow[r, "\sim", "\text{currying}"'] \arrow[d, "-\circ(g \otimes 1_A)"'] & \mathrm{Hom}(X, \mathrm{hom}(A,Y)) \arrow[d, "-\circ g"] \\ \mathrm{Hom}(X' \otimes A, Y) \arrow[r, "\sim", "\text{currying}"'] & \mathrm{Hom}(X', \mathrm{hom}(A,Y)) \end{tikzcd}$$

Let's use this to derive something about

$$ev : \hom(A,Y) \otimes A \longrightarrow Y$$

which was obtained by uncurrying

$$1 : \hom(A,Y) \longrightarrow \hom(A,Y)$$

We take $X = \hom(A,Y)$ & our square gives:

$$\begin{tikzcd}
ev \arrow[r, mapsto, "\text{currying}"'] \arrow[d, mapsto, "\circ(g\otimes 1_A)"'] & 1 : \hom(A,Y) \to \hom(A,Y) \arrow[d, mapsto, "\circ g"] \\
ev\circ(g\otimes 1_A) \arrow[r, mapsto, "\text{currying}"'] & \widetilde{ev\circ(g\otimes 1_A)} = g
\end{tikzcd}$$

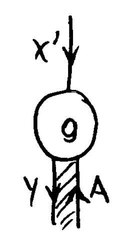
so we get

$$\widetilde{ev\circ(g\otimes 1_A)} = g$$

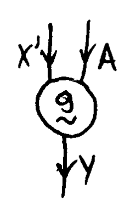
Uncurrying both sides we get

$$ev\circ(g\otimes 1_A) = \underset{\sim}{g}$$

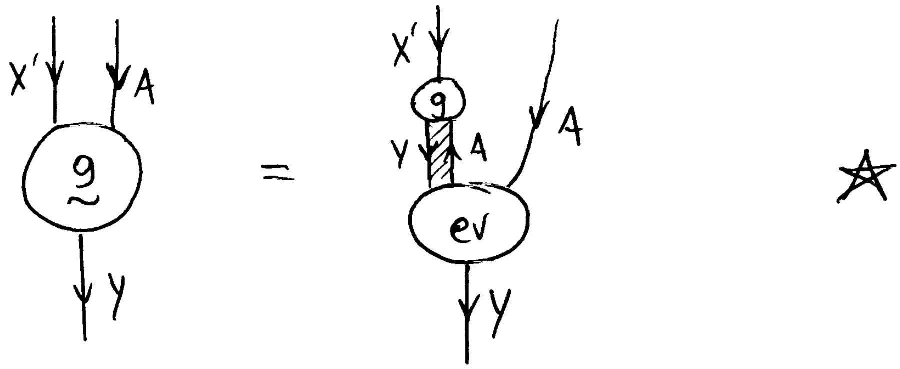
Let's draw this:

$g : X' \to \hom(A,Y)$ looks like 

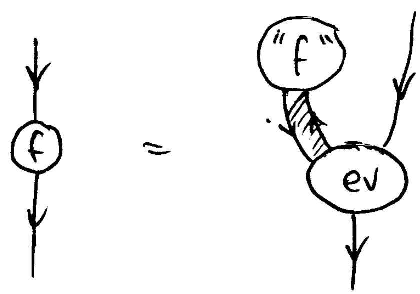
so

$\underset{\sim}{g} : X' \otimes A \to Y$ looks like 

and our equation says this equals $ev \circ (g \otimes 1_A)$:

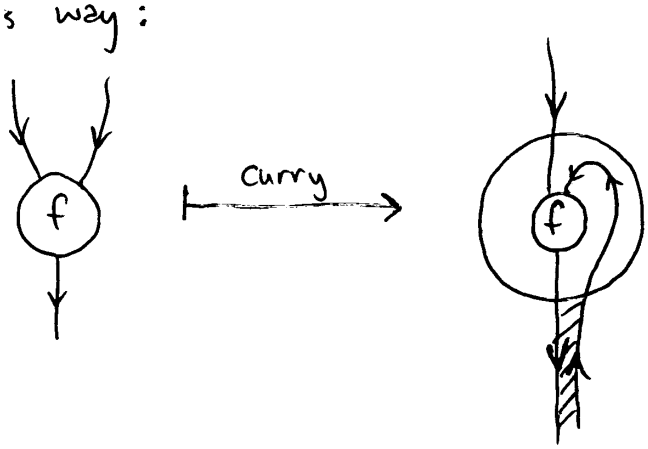


We were trying to prove



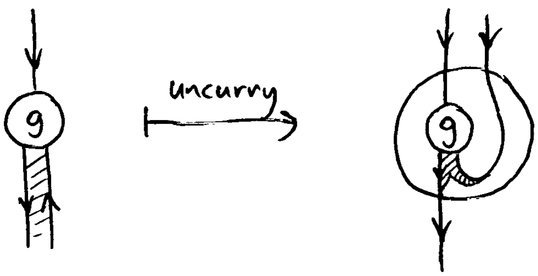
so we just take $X' = I$, $g = $ "f". ∎

The naturality equation looks nice if we draw currying this way:



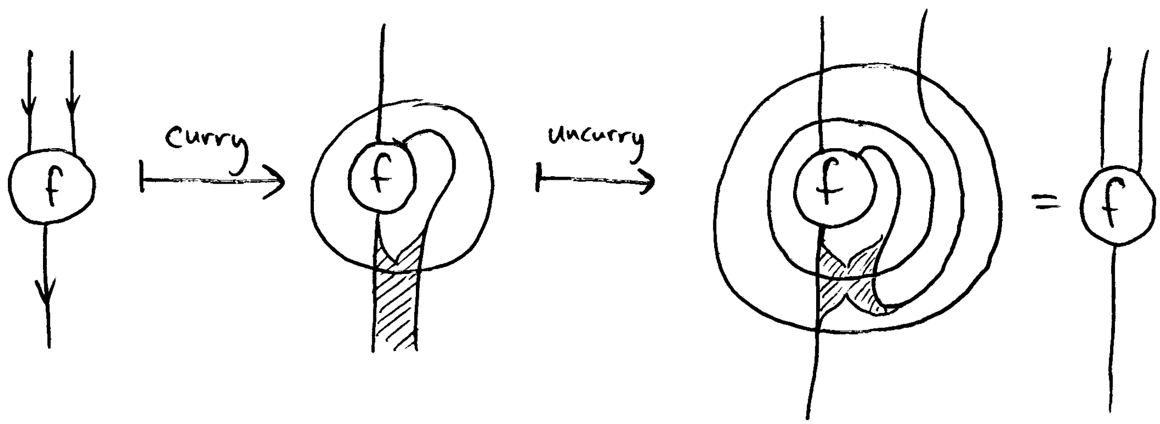
where the "bubble" indicates there may be no morphism $\curvearrowleft$.
We do have this if $\mathcal{C}$ is compact: $\hom(A,B) \cong B \otimes A^*$, e.g.

this works in $\mathcal{C} = \mathrm{Vect}$ (think: "quantum"!) but not $\mathcal{C} = \mathrm{Set}$ (think: "classical"!). So you can "pop" the bubble if your category is compact. Similarly, to uncurry:

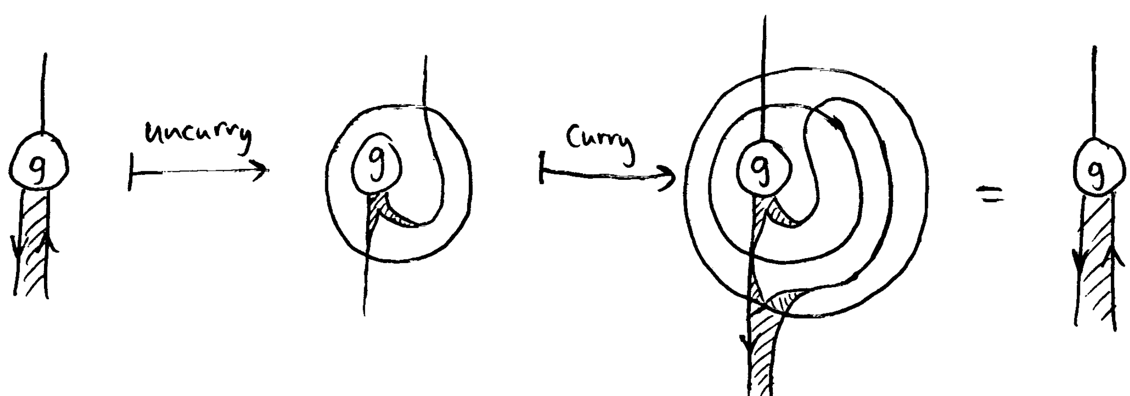


where only is a morphism in the compact case.

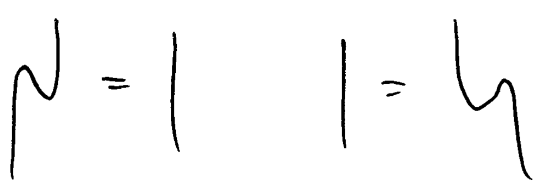
So:



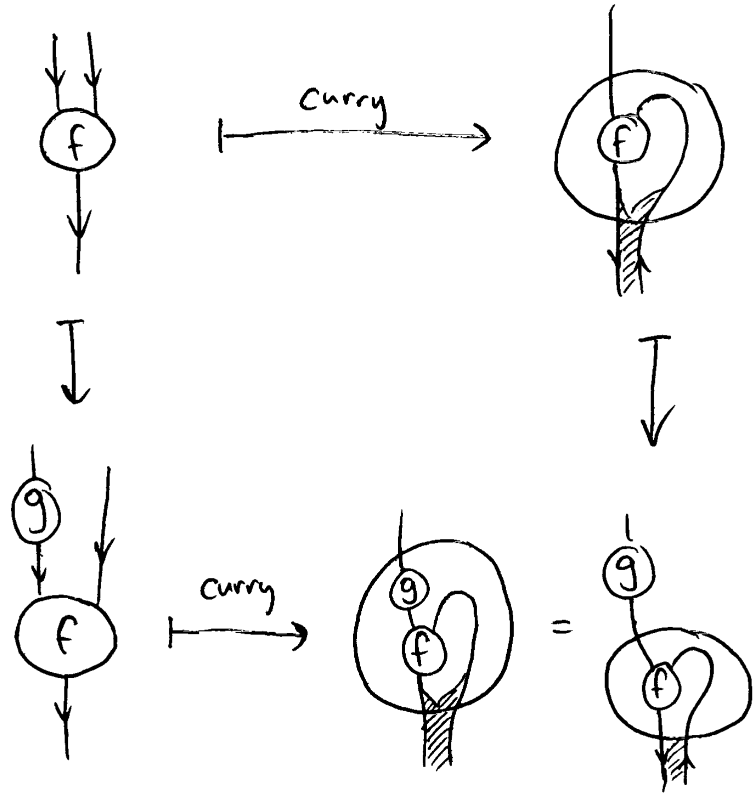
&



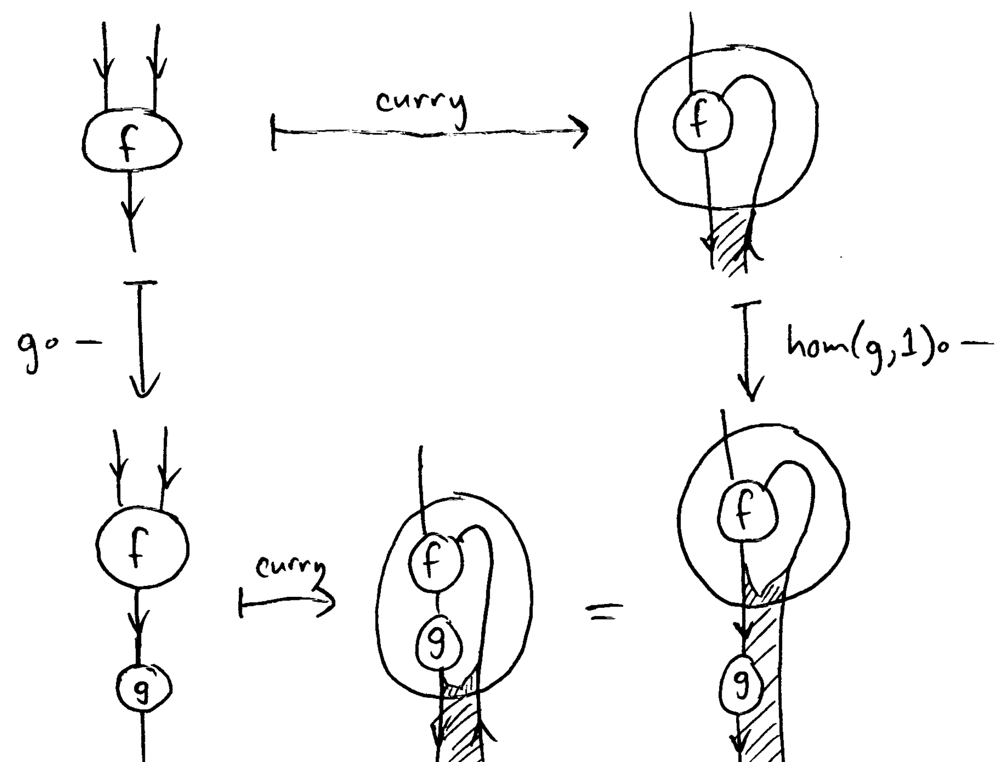
In a compact category, these follow from the zigzag identities



Our naturality equation ★ looks nice now:



Naturality also gives



So: the bubbles are "permeable".