The Canonical 1-form

We've seen that the canonical 1-form

$$\alpha = p_i \, dq^i$$

on T^*X plays 2 roles in classical mechanics:

1) $d\alpha = \omega$ lets you do Hamiltonian mechanics on the "phase space" T^*X.

2) The integral of α along a path

$$\gamma : [t_0, t_1] \longrightarrow T^*X$$

is almost the action of that path, which gives Lagrangian mechanics:

$$S(\gamma) = \int_{t_0}^{t_1} \overbrace{(p_i \dot{q}^i - H)}^{\text{Lagrangian}} \, dt$$

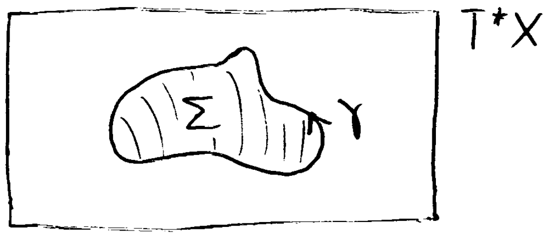
$$= \int_\gamma \alpha - \int_{t_0}^{t_1} H \, dt$$

$$= \int_\gamma \alpha - E(t_1) + E(t_0)$$

where the last step works if

$$\gamma : [t_0, t_1] \longrightarrow \{ H(q,p) = E \} \subseteq T^*X.$$

Putting these together we can give a physical interpretation of $\int_\Sigma \varpi$ where $\Sigma \subseteq T^*X$ is a surface with $\partial\Sigma = \gamma$ for some loop γ:



Stokes' Theorem says

$$\int_\Sigma \varpi = \int_\Sigma d\alpha = \int_{\gamma=\partial\Sigma} \alpha = S(\gamma) + \int_{t_0}^{t_1} H\, dt$$

which is just $S(\gamma)$ in the limit where we let $t_1 \to t_0$, reparameterizing γ suitably.

Moral: ϖ tells you the action it costs to run around the surface Σ.

Now let's go to the extended phase space

$$T^*(X \times \mathbb{R})$$

A point in the extended configuration space $(q,t) \in X \times \mathbb{R}$ says where & when your system is. Similarly, a point

$$\underset{\substack{\cap\\ X}}{q},\ \underset{\substack{\cap\\ \mathbb{R}}}{t},\ \underset{\substack{\cap\\ T_q^*X}}{p},\ \underset{\substack{\cap\\ T_t^*\mathbb{R}}}{p_0} \quad (q,t,p,p_0) \in T^*(X \times \mathbb{R})$$

specifies position, time, momentum and <u>energy</u> $E = -p_0$.

In these terms any path

$$\gamma : [t_0, t_1] \longrightarrow T^*X$$
$$t \longmapsto (q(t), p(t))$$

gives

$$\tilde{\gamma} : [t_0, t_1] \longrightarrow T^*(X \times \mathbb{R})$$
$$t \longmapsto (q(t), t, p(t), -H(q(t), p(t)))$$

if we choose a Hamiltonian $H : T^*X \longrightarrow \mathbb{R}$. Now the action is:

$$\begin{aligned} S(\gamma) &= \int_{t_0}^{t_1} \left(p_i \dot{q}^i - H(q(t), p(t))\right) dt \\ &= \int_{t_0}^{t_1} p_i\, dq^i - H(q(t), p(t))\, dt \\ &= \int_{\tilde{\gamma}} \tilde{\alpha} \end{aligned}$$

where

$$\tilde{\alpha} = p_i\, dq^i + p_0\, dt$$

is the canonical 1-form on $T^*(X \times \mathbb{R})$, since the p_0 coordinate of the path $\tilde{\gamma}$ was cleverly chosen to be $-H(q(t), p(t))$.

Carlo Rovelli (see http://arxiv.org/abs/gr-qc/0207043) has reformulated this idea to apply not just to particles, but strings and higher-dimensional "branes." This requires a canonical <u>2-form</u> (or higher form) on ... <u>something</u> (not the cotangent bundle, of course — there's no such thing!)

To understand this approach, let's allow $\tilde{\gamma}$ to be more general:

$$\tilde{\gamma} : [s_0, s_1] \longrightarrow T^*(X \times \mathbb{R})$$
$$s \longmapsto (q(s), t(s), p(s), -H(q(s), p(s)))$$

where $H : T^*X \to \mathbb{R}$ is as before. In other words

$$\tilde{\gamma} : [s_0, s_1] \longrightarrow Y \subseteq T^*(X \times \mathbb{R})$$

where Y contains all information about H:

$$Y = \{ (q, p, t, p_0) \; : \; p_0 = -H(q, p) \}$$

Y is a codimension 1 submanifold, therefore not symplectic, but it has a 1-form on it:

$$\tilde{\alpha}|_Y = i^*\tilde{\alpha}$$

— i.e. the pullback of $\tilde{\alpha}$ along the inclusion

$$i : Y \hookrightarrow T^*(X \times \mathbb{R})$$

We also have a 2-form on Y:

$$\tilde{\omega}\big|_Y = i^*\tilde{\omega}.$$

Claim: a curve

$$\tilde{\gamma} : [s_0, s_1] \longrightarrow Y$$

gives a solution of Hamilton's equations iff its tangent vector $\tilde{\gamma}'(s)$ ~~sa~~tisfies

$$\tilde{\omega}\big|_Y(\tilde{\gamma}'(s), -) = 0.$$

Note: even though $\tilde{\omega}$ is nondegenerate, $\tilde{\omega}|_Y$ is not, so this equation has nontrivial solutions.

$$\tilde{\gamma}'(s) = \frac{dq^i(s)}{ds}\frac{\partial}{\partial q^i} + \frac{dp_i(s)}{ds}\frac{\partial}{\partial p_i} + \frac{dt}{ds}\frac{\partial}{\partial t} + \underbrace{\frac{dp_0}{ds}}_{\displaystyle \parallel \atop \displaystyle -\frac{\partial H}{\partial q^i}\frac{dq^i}{ds} - \frac{\partial H}{\partial p_i}\frac{dp_i}{ds}}\frac{\partial}{\partial p_0}$$

Now compute

$$\tilde{\omega}|_Y(\tilde{\gamma}'(s), -) \quad \text{using} \quad \tilde{\omega} = dp_i \wedge dq^i + dp_0 \wedge dt$$

We get

$$\tilde{\omega}|_Y(\tilde{\gamma}'(s), -) = -\frac{dq^i}{ds}dp_i + \frac{dp_i}{ds}dq^i - \frac{dt}{ds}dp_0 - \left(\frac{\partial H}{\partial q^i}\frac{dq^i}{ds} + \frac{\partial H}{\partial p_i}\frac{dp_i}{ds}\right)dt$$

$$\Big\|_{\text{on } Y}$$

$$+\frac{dt}{ds}dH = +\frac{dt}{ds}\left(\frac{\partial H}{\partial q^i}dq^i + \frac{\partial H}{\partial p_i}dp_i\right)$$

This being zero is equivalent to

HAMILTON'S EQUATIONS

$$\left[\begin{array}{lll} \dfrac{dp_i}{ds} = \dfrac{dt}{ds}\dfrac{\partial H}{\partial q_i} & \underset{(\text{if } \frac{dt}{ds} \neq 0)}{\Longrightarrow} & \dfrac{dp_i}{dt} = \dfrac{\partial H}{\partial q_i} \\ \dfrac{dq^i}{ds} = -\dfrac{dt}{ds}\dfrac{\partial H}{\partial p_i} & \Longrightarrow & \dfrac{dq^i}{dt} = \dfrac{-\partial H}{\partial p^i} \end{array}\right.$$

CONSERVATION OF ENERGY

$$\left[\ \frac{\partial H}{\partial q^i}\frac{dq^i}{ds} + \frac{\partial H}{\partial p_i}\frac{dp_i}{ds} = 0 \quad \Longrightarrow \quad \frac{dH}{ds} = 0 \quad \Longrightarrow \quad \frac{dH}{dt} = 0\right.$$