From Particles to Strings and Membranes

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Now let's generalize everything we did so far from point particles:

$$\gamma : [s_0, s_1] \longrightarrow \underset{\text{'space'}}{X} \times \underset{\text{'time'}}{\mathbb{R}} = \underset{\text{'spacetime'}}{M}$$

to strings & membranes (or "branes")

$$\gamma : \Sigma \longrightarrow M$$

where now Σ is a p-dimensional manifold with boundary (or corners, like a cube: $\Sigma = [0,1]^p$). In the particle case, the canonical 1-form α on T^*M played a key role in defining the action

$$S(\tilde{\gamma}) = \int_{\tilde{\gamma}} \alpha$$

(using α to denote what we called $\tilde{\alpha}$ in the previous lecture)

where

$$\tilde{\gamma} : [s_0, s_1] \longrightarrow \underset{\{p_0 = -H(q,p)\}}{\overset{}{Y}} \subseteq T^*M$$

for our Hamiltonian $H : T^*X \to \mathbb{R}$. So <u>now</u> we need a "canonical p-form" so we can integrate it over our p-dimensional surface! (Pea-brains call such a thing a "(p-1)-brane".)

PARTICLES ($p=1$)

1-dimensional "worldline"

$$\gamma : [s_0, s_1] \longrightarrow M$$

Extended phase space T^*M

We start studying

$$\tilde{\gamma} : [s_0, s_1] \longrightarrow T^*M$$

because T^*M has a canonical 1-form on it.

Our worldline γ has a tangent vector

$$\gamma'(s) \in T_{\gamma(s)}M$$

called its <u>velocity</u>

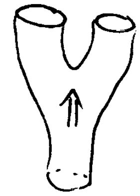
 $\in T_{\gamma(s)}M$

We can define γ' using the coordinate s on $[s_0, s_1]$:

$$\gamma'(s) = \frac{d\gamma(s)}{ds}$$

MEMBRANES ($p \geq 0$)

(note $p=0 \sim$ instantons)



p-dimensional membrane "worldvolume"

$$\gamma : \Sigma \longrightarrow M$$

(or "worldsheet" if $p=2$)

Extended phase space $\Lambda^p T^*M$

We start studying

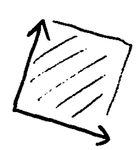
$$\tilde{\gamma} : \Sigma \longrightarrow \Lambda^p T^*M$$

because $\Lambda^p T^*M$ has a canonical p-form on it.

Our worldvolume γ has a tangent multivector ("p-vector")

$$\gamma'(x) \in \Lambda^p T_{\gamma(x)}M$$

called its <u>multivelocity</u> (or <u>p-velocity</u>), which says how fast & in which direction γ moves for each coordinate on Σ.

 $\in \Lambda^2 T_{\gamma(s)}M$

($p=2$)

Let's define γ' using local coordinates on Σ, say $s_1, \ldots, s_p$:

$$\gamma'(x) = \frac{\partial\gamma(x)}{\partial s_1} \wedge \cdots \wedge \frac{\partial\gamma(x)}{\partial s_p} \in \Lambda^p T_{\gamma(x)}M$$

(Warning: vanishes if the $\frac{\partial\gamma}{\partial s_i}$ are linearly dependent, i.e. γ not immersion)

One can also invent a coordinate-free approach to defining γ'

A Lagrangian was a function

$$L : T\underbrace{X}_{\text{space}} \longrightarrow \mathbb{R}$$

but we can consider a generalization where a Lagrangian is

$$L : \underbrace{TM}_{\text{spacetime}} \longrightarrow \mathbb{R}$$

This is good since now

$$(\underbrace{\gamma(s)}_{\text{position}}, \underbrace{\gamma'(s)}_{\text{velocity}}) \in TM$$

Challenge: can you invent a coordinate free definition of γ'?

A Lagrangian should be a function

$$L : \Lambda^p TM \longrightarrow \mathbb{R}$$

since now

$$(\gamma(x), \gamma'(x)) \in \Lambda^p TM$$