28 November 2006

From particles to Membranes (cont.)

Here's a way to define the "multivelocity" of a map

$$\gamma : \Sigma \longrightarrow M$$

(Σ: p-dim manifold; M: manifold)

i.e. a p-dimensional membrane in the spacetime M

$p = 1$ $\qquad\qquad$ $p = 2$

To define γ' in a coordinate-free way, take

$$\gamma : \Sigma \longrightarrow M$$

and form

$$d\gamma : T\Sigma \longrightarrow TM$$

or to be cute,

$$T\gamma : T\Sigma \longrightarrow TM$$

where T is a functor

$$T : [\text{manifolds}] \longrightarrow [\text{vector bundles}].$$

Then to get γ' apply the functor

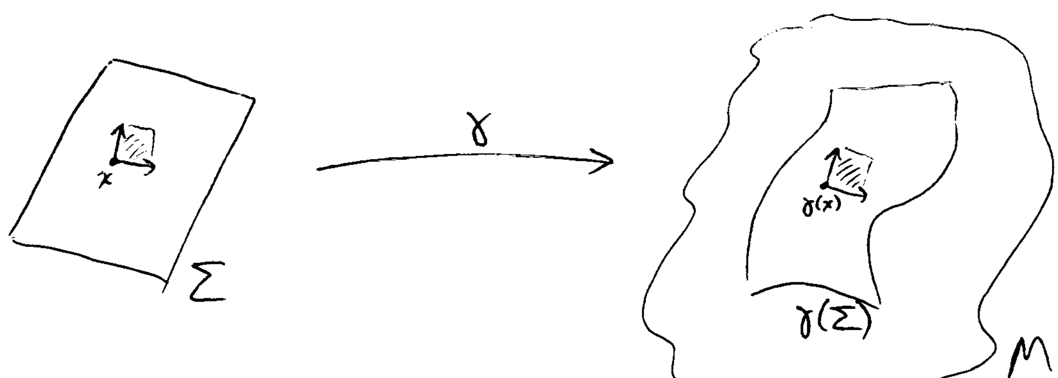
$$\Lambda^p : [\text{vector bundles}] \longrightarrow [\text{vector bundles}]$$

Given any vector bundle $E \to M$, $\Lambda^p E \to M$ is a new vector bundle with $(\Lambda^p E)_x = \Lambda^p(E_x)$ $\forall x \in M$.
Given any vector bundle map $f: E \to E'$, $\Lambda^p f : \Lambda^p E \to \Lambda^p E'$ is given by

$$(\Lambda^p f)_x(e_1 \wedge \cdots \wedge e_p) = f(e_1) \wedge \cdots \wedge f(e_p) \qquad \forall e_i \in E_x$$

So:

$$\Lambda^p d\gamma : \Lambda^p T_x \Sigma \longrightarrow \Lambda^p T_{\gamma(x)} M$$



If Σ is equipped with a volume form, i.e.

$$\mathrm{vol} \in \Omega^p(\Sigma)$$

s.t.

$$0 \neq \mathrm{vol}_x \in \Lambda^p T^* \Sigma.$$

then we get a unique

$$u \in \Lambda^p T_x \Sigma \quad \text{s.t. } \mathrm{vol}(u) = 1$$

since $\Lambda^p T_x \Sigma$ is one-dimensional.

Then define

$$\gamma'(x) = \Lambda^p d\gamma(u) \in \Lambda^p T_{\gamma(x)} M$$

PARTICLES ($p=1$)

A particle has a worldline

$$\gamma : [s_0, s_1] \longrightarrow M$$

In general relativity, M is a Lorentzian manifold & the action for a particle with mass m and charge q is:

$$S(\gamma) = m \cdot \mathrm{Length}(\gamma) + e \cdot \int_\gamma A$$

where $A \in \Omega^1(M)$ is the electromagnetic potential and

$$\mathrm{Length}(\gamma) = \int_{s_0}^{s_1} |\gamma'| \, ds$$

where $|\cdot|$ is defined using the metric. Note:

$$\int_{s_0}^{s_1} |\gamma'| \, ds$$

is independent of parametrization, since a parameterization that moves twice as fast is over in half the time.

MEMBRANES ($p \geq 0$)

A membrane has a worldvolume

$$\underset{\text{p-dim.}}{\gamma : \Sigma} \longrightarrow M$$

The usual Nambu-Goto action for a membrane is:

$$S(\gamma) = m \cdot \mathrm{Volume}(\gamma) + e \int_\gamma A$$

where m is the membrane tension, e is the charge and $A \in \Omega^p(M)$ is the p-form electromagnetic potential — the analog of the electromagnetic potential used in 'p-form electromagnetism'. Here

$$\mathrm{Volume}(\gamma) = \int_\Sigma |\gamma'| \, \mathrm{vol}$$

where we use the chosen volume form on Σ, & the metric on M gives an inner product on $\Lambda^p TM$, which we use to define $|\gamma'|$. Note $\int_\Sigma |\gamma'| \, \mathrm{vol}$ is independent of vol, since doubling vol halves γ'.

If $A=0$, a particle extremizing the action traces out a geodesic

If we consider only the electromagnetic part of the action:

$$e\int_\gamma A$$

this gives a change of phase

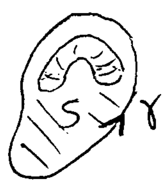
$$\exp\left(ie\int_\gamma A\right) \in U(1)$$

when we move a quantum particle of charge e along γ. In fact

$$\int_\gamma A = \int_S F$$

if

$$\partial S = \gamma$$



& $F=dA$ is the electromagnetic field.

If $A=0$, a membrane extremizing the action traces out a "minimal surface".

If we consider only the 'p-form electromagnetic' part of the action

$$e\int_\gamma A$$

we get a change of phase

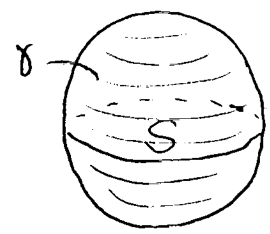
$$\exp\left(ie\int_\gamma A\right) \in U(1)$$

when we move a quantum membrane along the surface γ. In fact

$$\int_\gamma A = \int_S F$$

if

$$\partial S = \gamma$$



& $F=dA$ is the p-form electromagnetic field.