5 Dec. 2006

A Glimpse of What's to Come

Having reviewed classical particle mechanics and its stringy and brany generalizations, next we'll quantize these. We'll mainly talk about geometric quantization.

In particle mechanics, a phase space is often a symplectic manifold (M, ω). The role of the 2-form ω is to (help) compute the action for a path γ in M:

$$S(\gamma) = \int_\gamma \alpha$$

where $d\alpha = \omega$ (where now we drop the conventional minus sign). What about ω itself, though? If

$$\gamma = \partial \Sigma$$

we say γ is homologous to zero and then

$$S(\gamma) = \int_\Sigma \omega$$

But: 1) what if $\gamma \neq \partial\Sigma$ for any Σ?
2) what if $\omega \neq d\alpha$ for any α?

We get in trouble if both of these happen. But there's a generalization of a 1-form which can still work — a connection on a $U(1)$ bundle.

In quantum mechanics, what matters is not the action $S(\gamma) \in \mathbb{R}$ but the phase $e^{iS(\gamma)} \in U(1)$

$$0 \longrightarrow 2\pi\mathbb{Z} \longrightarrow \mathbb{R} \xrightarrow{s \mapsto e^{is}} U(1) \longrightarrow 0$$

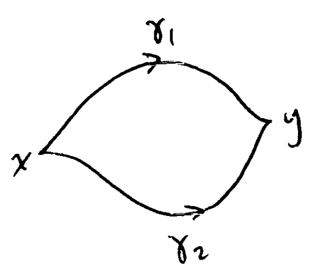
Really, in quantum mechanics, if you have a bunch of paths from $x \in M$ to $y \in M$, the amplitude to get from x to y is

$$A = \int_{\substack{\{\gamma : [s_0, s_1] \to M \\ \gamma(s_0) = x \ \gamma(s_1) = y\}}} e^{iS(\gamma)} \, D\gamma \quad \in \mathbb{C}$$

a path integral. The amplitude gives a probability $|A|^2$ (up to normalization). So for physics, doing

$$e^{iS(\gamma)} \longmapsto c\, e^{iS(\gamma)}$$

for some $c \in U(1)$ doesn't change any probabilities. But a connection on a $U(1)$ bundle lets you calculate a definite phase for any loop γ, and thus a ratio of phases for any pair of paths γ_1 & γ_2 from x to y. Why?



$$\frac{\text{Phase}(\gamma_1)}{\text{Phase}(\gamma_2)} = \text{Phase}(\gamma_1 \gamma_2^{-1})$$

a loop

These relative phases are what we can actually measure: they are unchanged by multiplying $\mathrm{Phase}(\gamma_1)$ & $\mathrm{Phase}(\gamma_2)$ by any $c \in U(1)$. Any $U(1)$ connection has a curvature 2-form $\varpi \in \Omega^2(M)$, which should be our symplectic structure in applications to geometric quantization. If γ is a loop with $\partial\Sigma = \gamma$ for some surface Σ:

$$\mathrm{Phase}(\gamma) = e^{i\int_\Sigma \varpi}$$

This is just like

$$e^{i\int_\gamma A} = e^{i\int_\Sigma \varpi}$$

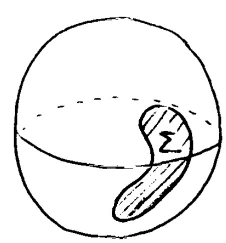
But now our "$\mathrm{Phase}(\gamma)$" is not coming from a 1-form A on M anymore.

Example: $M = S^2$, unit sphere with

ϖ

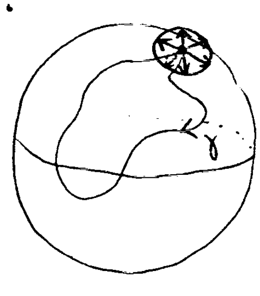
so that

$$\int_\Sigma \varpi = \mathrm{Area}(\Sigma)$$



What's our $U(1)$ connection?

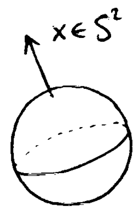
Given a loop γ, pick a unit tangent vector v an parallel transport it around γ:



In this case we really do have

$$\text{Phase}(\gamma) = e^{i\int_\Sigma \omega} = e^{i\,\text{Area}(\Sigma)}$$

when $\partial\Sigma = \gamma$. We say ω is the curvature of our $U(1)$ connection. M is the phase space of the <u>rigid rotor</u> – a rigid spinning ball w. fixed rate of rotation.



This is also the Riemann sphere. (This gives an example of a phase space that is not a cotangent bundle). There's no 1-form on M with $d\alpha = \omega$. If there were, & $\partial\Sigma = \gamma$, then

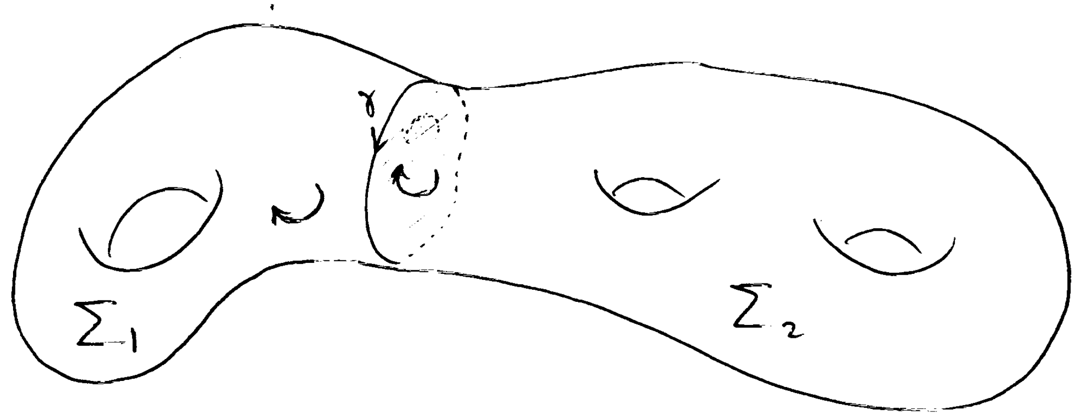
$$\int_\Sigma \omega = \int_\gamma \alpha$$

by Stokes' theorem, since if $\Sigma = S^2$ then $\partial\Sigma = \emptyset$ so we get $\text{Area}(S^2) = \int_\Sigma \omega = \int_\gamma \alpha = 0$.

More generally, say (M, ω) is any symplectic manifold equipped with a $U(1)$ connection whose curvature is ω:

$$\text{Phase}(\gamma) = e^{i\int_\Sigma \omega} \qquad \text{when } \partial\Sigma = \gamma.$$

In this case γ might be the boundary of several interesting surfaces:



Then we have

$$\text{Phase}(\gamma) = e^{i\int_{\Sigma_1}\varpi}$$
$$= e^{-i\int_{\Sigma_2}\varpi}$$

so if $\Sigma = \Sigma_1 \cup \Sigma_2$ then

$$e^{i\int_\Sigma \varpi} = e^{i\int_{\Sigma_1}\varpi + i\int_{\Sigma_2}\varpi} = 1.$$

So

$$\int_\Sigma \varpi \in 2\pi\mathbb{Z} \quad !$$

in fact, for any $\Sigma \subseteq M$ with $\partial\Sigma = \emptyset$, since we can always write $\Sigma_1 \cup \Sigma_2$. So: ϖ describes an integral 2nd cohomology class. This is one origin of the term "quantization" – the allowed ϖ's are quantized.

All this stuff generalizes to $(p+1)$-forms replacing our 2-form ϖ!