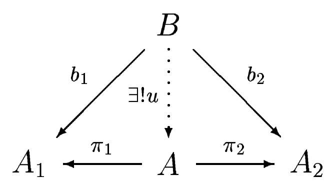
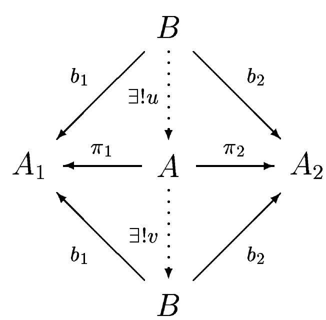
QG F06 Homework 2
Mike Stay

1. *Show that every category with finite products can be made into a monoidal category.*

Let $\mathcal{C}$ be a category with finite products. Then for any pair of objects A_1, A_2, there exists another object A and morphisms $\pi_1 : A \to A_1$, $\pi_2 : A \to A_2$ such that for any other object B with maps $b_1 : B \to A_1$, $b_2 : B \to A_2$ there exists a unique map $u : B \to A$ such that

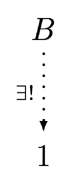


commutes. If B is another product, then we have a unique $v : A \to B$ such that

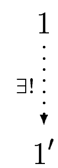


commutes. Since $u \circ v : B \to B$ is unique and $1_B : B \to B$ exists, then $u \circ v = 1_B$. By symmetry, $v \circ u = 1_A$, thus $A \cong B$ and any product $A_1 \times A_2$ we choose will be isomorphic to any other.

Let 1 be an object of $\mathcal{C}$ such that for any object B there exists a unique map



Define $I := 1$. This is well defined up to isomorphism, since for any other choice $1'$ there exists a unique isomorphism such that



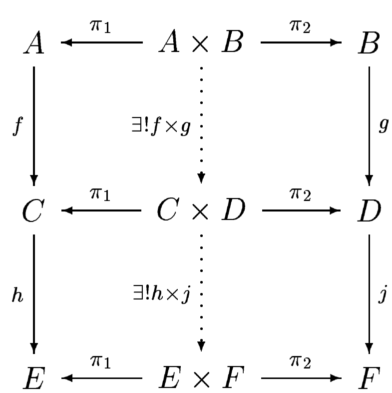
commutes.

Define $A \otimes B := A \times B$. Now $\mathcal{C}$ is a monoidal category.

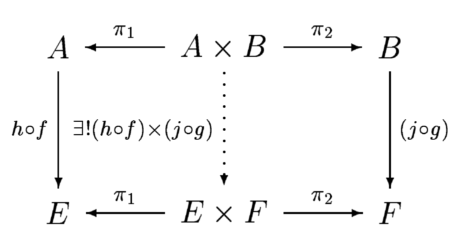
- The map

$$\begin{aligned} \otimes : \mathbb{C} \times \mathbb{C} &\to \mathbb{C} \\ (A, B) &\mapsto A \otimes B \\ (f, g) &\mapsto f \otimes g \end{aligned}$$

  preserves composition, since given $f : A \to C$, $g : B \to D$, $h : C \to E$, $j : D \to F$, we have unique morphisms $f \otimes g : A \otimes B \to C \otimes D$ and $h \otimes j : C \otimes D \to E \otimes F$ such that this diagram commutes:

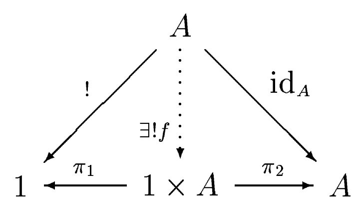


  But this diagram also commutes:

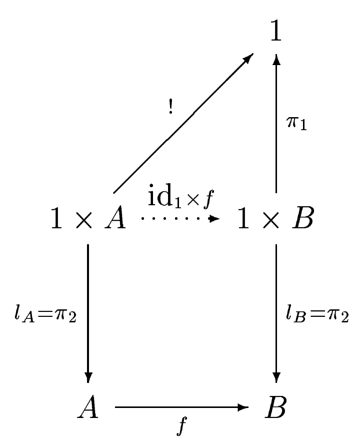


  so $(h \otimes j) \circ (f \otimes g) = (h \circ f) \otimes (j \circ g)$. Since $\otimes$ is defined for objects and morphisms and preserves composition, $\otimes$ is a functor.

- $A \cong I \otimes A$ since there exists a unique f making the following commute

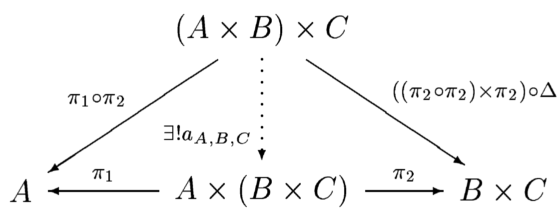


  and $f \circ pi_2 = \pi_2 \circ f = \text{id}_A$. A similar argument shows $A \cong A \otimes I$.
- The unitors $r_A : A \otimes 1 \to A = \pi_1$ and $l_A : 1 \otimes A \to A = \pi_2$ are natural isomorphisms. For l_-, consider the functors $\text{id}_\mathcal{C}$ and $1 \otimes -$. Given any morphism $f : A \to B$, we have a unique morphism $\text{id}_I \otimes f : I \otimes A \to I \otimes B$ making this diagram commute:

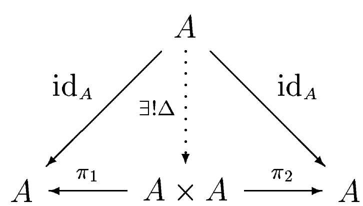


  and similarly for r_A.

- The associator $a_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ is the unique map making the following diagram commute:

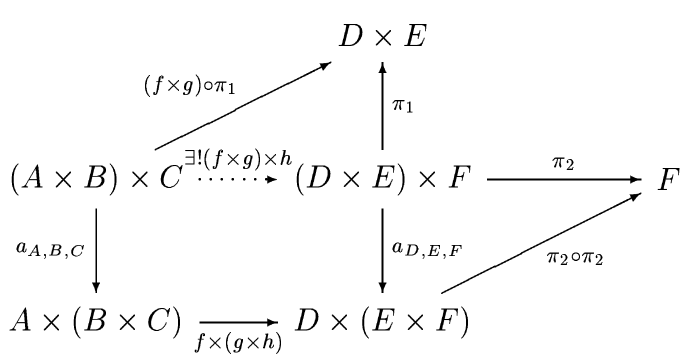


where $\Delta : A \to A \times A$ is the unique map making the following diagram commute:



A similar diagram shows there's a unique map from $A \otimes (B \otimes C)$ to $(A \otimes B) \times C$ with the appropriate property, so $a_{A,B,C}$ is an isomorphism.

- The associator is a natural isomorphism: given $f : A \to D$, $g : B \to E$, and $h : C \to F$, consider the two functors $(- \otimes -) \otimes - : \mathcal{C}^3 \to \mathcal{C}$, $- \otimes (- \otimes -) : \mathcal{C}^3 \to \mathcal{C}$.



- The "triangle equation" diagram and the "pentagon equation" diagrams commute for the same reason: since there are projections mapping out of each product and any map between products is unique, any two apparent ways of mapping from one product to another while preserving the components of the product must be the same.