J. Prasad Senesi

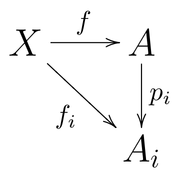
Quantum Gravity Seminar
Fall 2006

Homework 2
October 12, 2006

Suppose $\mathfrak{C}$ is a category with finite products; i.e., given objects $A_1, \ldots, A_n$ ($n \geq 0$) there exists an object A with morphisms

$$p_i : A \to A_i$$

such that for any $f_i : X \to A_i$, there exists a unique map $f : X \to A$ such that



commutes for all i.

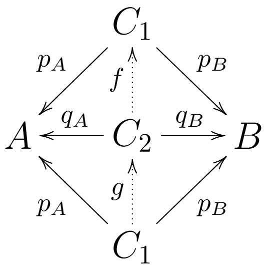
Show we can make $\mathfrak{C}$ into a monoidal category by choosing a product $A \times B$ for any pair of objects and by choosing a terminal object (a product of *no* objects) 1 and defining $I = 1$.

1 Defining the Tensor product

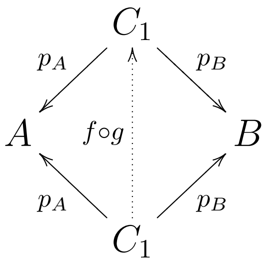
1.1 Uniqueness of products

Proposition. *Let A, B be objects in $\mathfrak{C}$, and C_1, C_2 be products of A and B. Then $C_1 \cong C_2$.*

Proof. Let p_a, p_b be the projection maps from C_1 to A, B, respectively, and q_a, q_b be the projection maps from C_2 to A, B. Consider the following diagram:



The maps $C_2 \xrightarrow{f} C_1$ and $C_1 \xrightarrow{g} C_2$ are those that are guaranteed to exist uniquely by the universal product properties of C_1 and C_2, respectively. Therefore there exists a unique map $f \circ g$ such that the following diagram commutes:

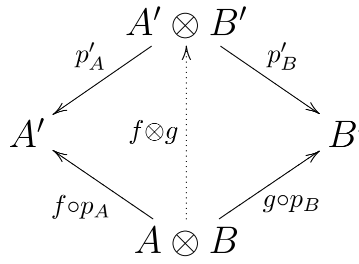


But the identity map $C_1 \xrightarrow{1_{C_1}} C_1$ will also satisfy this diagram in place of $f \circ g$, and therefore we must have $f \circ g = 1_{C_1}$. A similar argument will show that $g \circ f = 1_{C_2}$. Therefore $C_1 \cong C_2$. □

1.2 Definition of the functor ⊗

Using the axiom of choice, for every object $(A, B) \in \mathfrak{C} \times \mathfrak{C}$ we can choose one product in $\mathfrak{C}$ among the isomorphism class of products of A and B. Denote this chosen product $A \otimes B$.

If $(f, g) \in \hom((A, B), (A', B'))$, denote by $f \otimes g$ the unique map such that the diagram



commutes.

Proposition. *Define a map* $\otimes : \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$ *by*

$$\begin{aligned} \otimes : (A, B) &\mapsto A \otimes B \\ \otimes : (f, g) &\mapsto f \otimes g \end{aligned}$$

Then F is a (covariant) functor.

Proof. By definition we have

$$(A, B) \xrightarrow{(f,g)} (A', B') \Rightarrow A \otimes B \xrightarrow{f \otimes g} A' \otimes B'.$$

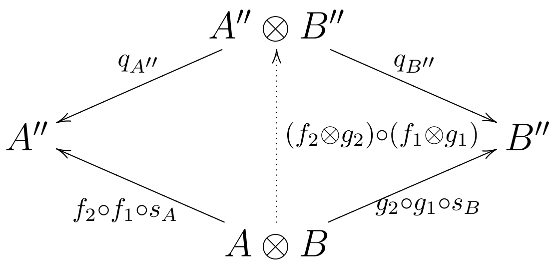
Let $A \otimes B \xrightarrow{f_1 \otimes g_1} A' \otimes B' \xrightarrow{f_2 \otimes g_2} A'' \otimes C''$, with projection maps

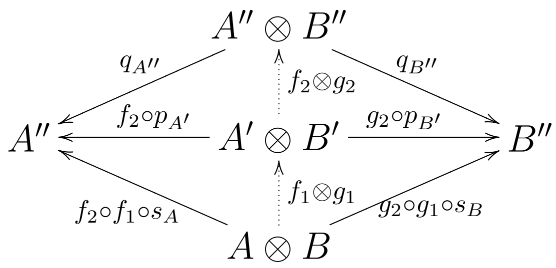
$$A \xleftarrow{s_A} A \otimes B \xrightarrow{s_B} B$$

$$A' \xleftarrow{p_{A'}} A' \otimes B' \xrightarrow{p_{B'}} B'$$

$$A'' \xleftarrow{q_{A''}} A'' \otimes B'' \xrightarrow{q_{B''}} B''.$$

Then both of the following diagrams commute:

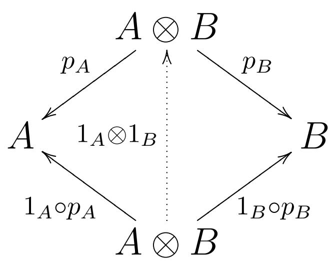




Therefore we must have

$$\otimes(f_2, g_2) \circ \otimes(f_1, g_1) = (f_2 \otimes g_2) \circ (f_1 \otimes g_1) = (f_2 \circ f_1) \otimes (g_2 \circ g_1) = \otimes((f_2, g_2) \circ (f_1, g_1)).$$

$1_A \otimes 1_B$ is defined to be the unique map which provides commutativity of this diagram:



But $1_{A \otimes B}$ also satisfies this criterion, and therefore

$$\otimes((1_A, 1_B)) = 1_A \otimes 1_B = 1_{A \otimes B} = 1_{\otimes((A,B))}.$$

□

Remark. Unless otherwise noted, for objects A and B we will denote the projection maps from $A \otimes B$ to A and B by p_A and p_B, respectively.

Proposition.

1. *Let A, B, C be objects in $\mathfrak{C}$. Then $(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$ are both products for A, B, C.*
2. *Let A, B, C, D be objects in $\mathfrak{C}$. Then*

$$A \otimes (B \otimes (C \otimes D)),$$
$$A \otimes ((B \otimes C) \otimes D),$$
$$(A \otimes (B \otimes C)) \otimes D,$$
$$(A \otimes B) \otimes (C \otimes D),$$
$$((A \otimes B) \otimes C) \otimes D$$

are all products for A, B, C, D.

The proof of this proposition is omitted. Its generalization to products of n objects for all $n > 0$ is clear.

1.3 The terminal object of $\mathfrak{C}$

Let us now fix a product of 0 objects in $\mathfrak{C}$, and denote this object I. Then, since for any object A in $\mathfrak{C}$ there exists a unique map t_A such that the diagram

$$A \xrightarrow{t_A} I$$

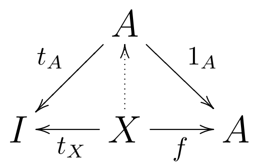
commutes, we see that I is a terminal object in $\mathfrak{C}$.

2 Unity and Associativity

2.1 The Left and Right Unitors

Lemma. *Let A be an object in $\mathfrak{C}$. Then $A, I \otimes A$, and $A \otimes I$ are all products of I and A.*

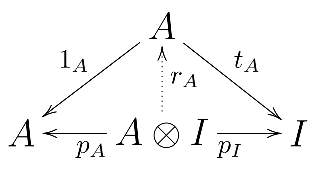
Proof. We will only prove that A is a product for I and A. Let X be an object of $\mathfrak{C}$, and $X \xrightarrow{f} A$. Then f is the unique morphism which allows the diagram



to commute. □

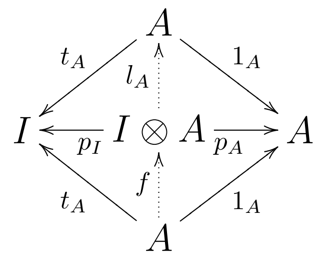
Therefore there exist unique morphisms $I \otimes A \xrightarrow{l_A} A$ and $A \otimes I \xrightarrow{r_A} A$. Let us call these morphisms the *left and right unitors of* A.

Using A as a product of A and I, we can see by the diagram



that $p_a = r_a$, and similarly for the projection map $I \otimes A \xrightarrow{p_A} A$, we have

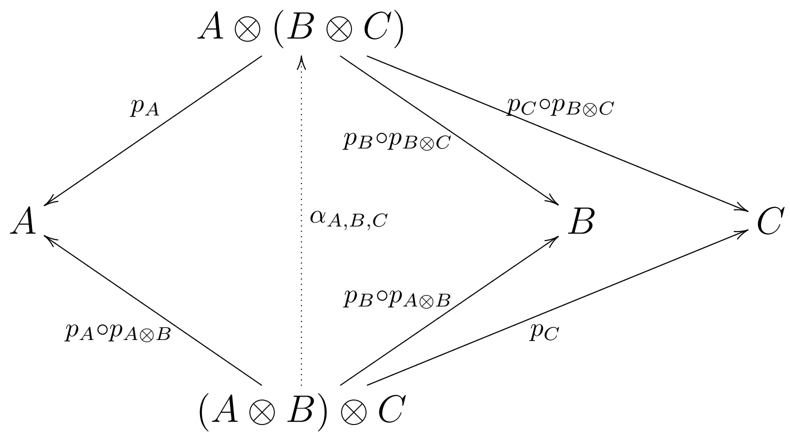
$p_A = l_A$. Furthermore, we can see from the diagram



that we must have $l_A \circ f = 1_A$, and a similar diagram shows that l_A is also left-invertible. Therefore l_A (and similarly r_A) is an isomorphism.

2.2 The Associators

As mentioned above, for any three objects A, B, C in $\mathfrak{C}$, $(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$ are both products. Therefore there exists a unique morphism $\alpha_{A,B,C}$ such that the following diagram commutes:



The identities satisfied by these morphisms are recorded here, for future use:

$$\begin{aligned} p_A \circ p_{A \otimes B} &= p_A \circ \alpha_{A,B,C} \\ p_B \circ p_{A \otimes B} &= p_B \circ p_{B \otimes C} \circ \alpha_{A,B,C} \\ p_C &= p_C \circ p_{B \otimes C} \circ \alpha_{A,B,C} \end{aligned}$$

3 Naturality

The goal of this section is to prove that l_A, r_A, and $\alpha_{A,B,C}$ are all natural isomorphisms.

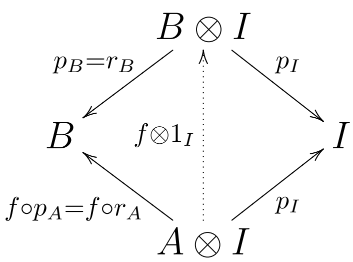
3.1 The functor $F : A \mapsto A \otimes I$

Define a functor from $\mathfrak{C}$ to $\mathfrak{C}$ as follows: If A is an object of $\mathfrak{C}$ and $A \xrightarrow{f} B$,

$$F : \begin{cases} A & \mapsto r_A^{-1}(A) = A \otimes I \\ f & \mapsto f \otimes 1_I. \end{cases}$$

Proposition. *Let* $\mathbf{I}$ *be the identity functor on* $\mathfrak{C}$*. Then* $\mathbf{I}$ *and* F *are naturally isomorphic.*

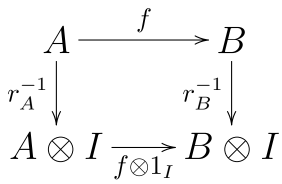
Proof. Let $A \xrightarrow{f} B$. Recall that the morphism $f \otimes 1_I$ is the unique morphism such that the diagram



commutes. Therefore we have the relation

$$r_B \circ (f \otimes 1_I) = f \circ r_A.$$

Inverting this relation on both sides by r_A, r_B gives us commutativity of the square



which provides a natural isomorphism $\mathbf{I} \Rightarrow F$. □

Corollary. *The left and right unitors are natural isomorphisms.*

3.2 The functor $G : (A \otimes B) \otimes C \mapsto A \otimes (B \otimes C)$

Define the functor G from $\mathfrak{C}$ to $\mathfrak{C}$ as follows: for objects A, B, C and morphisms $A \xrightarrow{f} A', B \xrightarrow{g} B'$, and $C \xrightarrow{h} C'$,

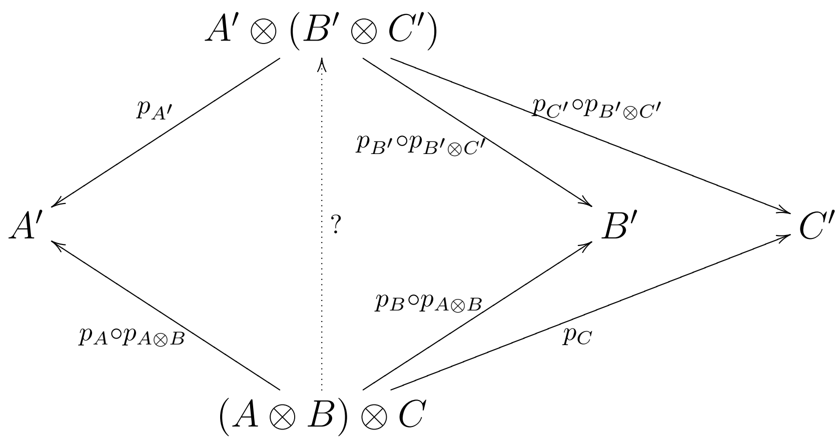
$$G : \begin{cases} (A \otimes B) \otimes C & \mapsto \quad \alpha_{A,B,C}((A \otimes B) \otimes C) = A \otimes (B \otimes C) \\ (f \otimes g) \otimes h & \mapsto \quad \alpha_{A,B,C}((f \otimes g) \otimes h) = f \otimes (g \otimes h) \end{cases}$$

Proposition. *The functors* $\mathbf{I}$ *and* G *are naturally isomorphic.*

Proof. We must show that the following square commutes:

$$\begin{array}{ccc} (A \otimes B) \otimes C & \xrightarrow{(f\otimes g)\otimes h} & (A' \otimes B') \otimes C' \\ \downarrow{\scriptstyle \alpha_{A,B,C}} & & \downarrow{\scriptstyle \alpha_{A',B',C'}} \\ A(\otimes B \otimes C) & \xrightarrow{f\otimes(g\otimes h)} & A'(\otimes B' \otimes C') \end{array}$$

We will do this by showing that both routes $\alpha_{A',B',C'} \circ (f \otimes g) \otimes h$ and $f \otimes (g \otimes h) \circ \alpha_{A,B,C}$ along the square from $(A \otimes B) \otimes C$ to $A'(\otimes B' \otimes C')$ provide commutativity of the following diagram:



Since only one map can do this, the result follows. For the following calculations, we will freely use the identities among morphisms provided by the commutative diagrams in sections 1.2 and 2.2. First we will show that $f \otimes (g \otimes h) \circ \alpha_{A,B,C}$ commutes.

$$
\begin{aligned}
p_{A'} \circ f \otimes (g \otimes h) \circ \alpha_{A,B,C} &= f \circ p_A \circ \alpha_{A,B,C} \\
&= f \circ p_A \circ p_{A \otimes B}.
\end{aligned}
$$

$$
\begin{aligned}
p_{B'} \circ p_{B' \otimes C'} \circ f \otimes (g \otimes h) \circ \alpha_{A,B,C} &= p_{B'} \circ (g \otimes h) \circ p_{B \otimes C} \circ \alpha_{A,B,C} \\
&= g \circ p_B \circ p_{B \otimes C} \circ \alpha_{A,B,C} \\
&= g \circ p_B \circ p_{A \otimes B}.
\end{aligned}
$$

$$
\begin{aligned}
p_{C'} \circ p_{B' \otimes C'} \circ f \otimes (g \otimes h) \circ \alpha_{A,B,C} &= p_{C'} \circ (g \otimes h) \circ p_{B \otimes C} \circ \alpha_{A,B,C} \\
&= h \circ p_C \circ p_{B \otimes C} \circ \alpha_{A,B,C} \\
&= h \circ p_C.
\end{aligned}
$$

Now we show that $\alpha_{A',B',C'} \circ (f \otimes g) \otimes h$ commutes as well:

$$
\begin{aligned}
p_{A'} \circ \alpha_{A',B',C'} \circ (f \otimes g) \otimes h &= p_{A'} \circ p_{A' \otimes B'} \circ (f \otimes g) \otimes h \\
&= p_{A'} \circ (f \otimes g) \circ p_{A \otimes B} \\
&= f \circ p_A \circ p_{A \otimes B}
\end{aligned}
$$

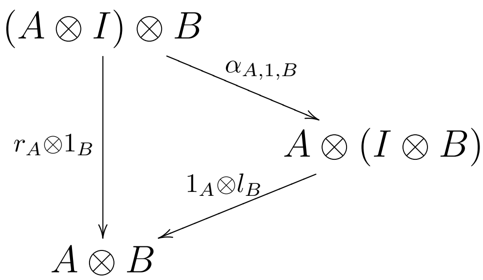
$$
\begin{aligned}
p_{B'} \circ p_{B' \otimes C'} \circ \alpha_{A',B',C'} \circ (f \otimes g) \otimes h &= p_{B'} \circ p_{A' \otimes B'} \circ (f \otimes g) \otimes h \\
&= p_{B'} \circ (f \otimes g) \circ p_{A \otimes B} \\
&= g \circ p_B \circ p_{A \otimes B}
\end{aligned}
$$

$$
\begin{aligned}
p_{C'} \circ p_{B' \otimes C'} \circ \alpha_{A',B',C'} \circ (f \otimes g) \otimes h &= p_{C'} \circ (f \otimes g) \otimes h \\
&= h \circ p_C.
\end{aligned}
$$

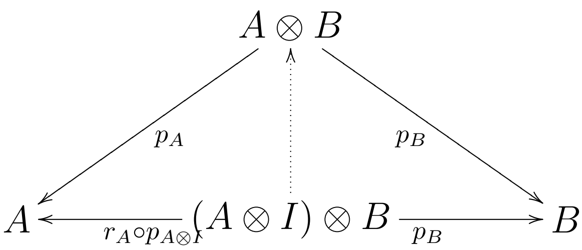
□

4 Triangle Identity of the Unitors

In order for $\mathfrak{C}$ to be a monoidal category, the following diagram must commute for all objects A, B in $\mathfrak{C}$:



We will show that $r_A \otimes 1_B = (1_A \otimes l_B) \circ \alpha_{A,I,B}$ by showing that both morphisms provide commutativity of the following diagram:



We have

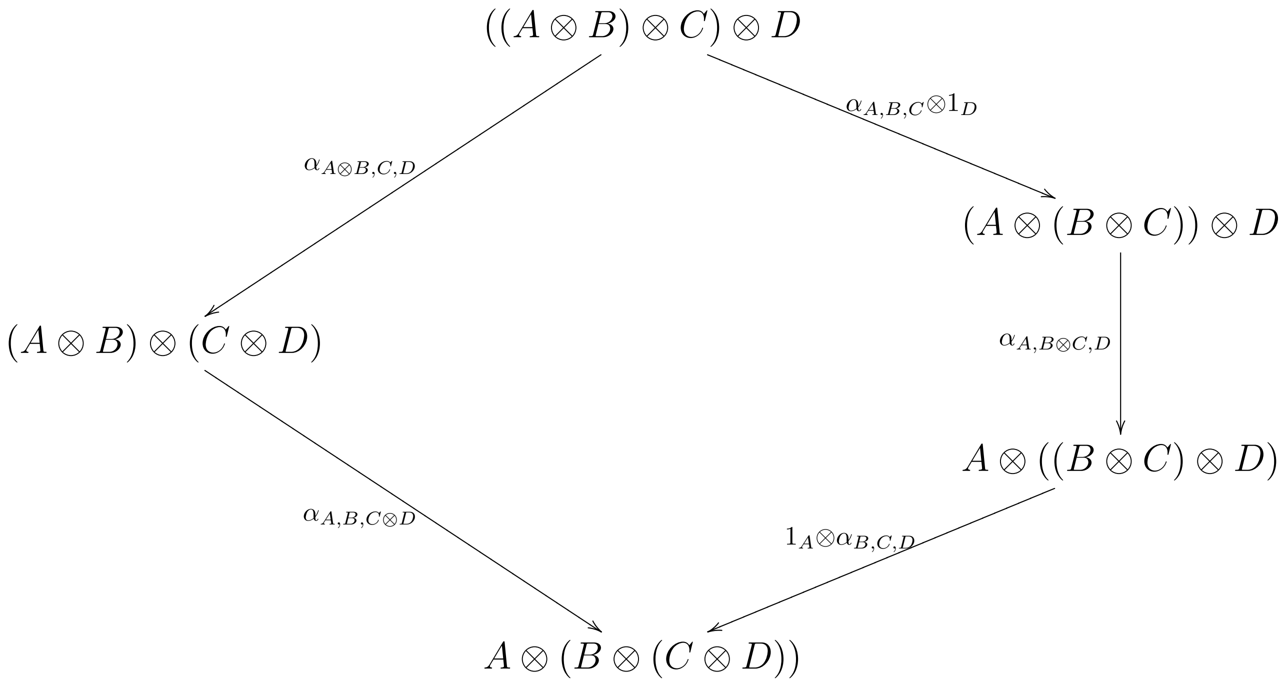
$$\begin{aligned} p_A \circ r_A \otimes 1_B &= r_A \otimes p_{A \otimes I} \\ \\ p_B \circ r_A \otimes 1_B &= p_B, \end{aligned}$$

and

$$\begin{aligned} p_A \circ (1_A \otimes l_B) \circ \alpha_{A,I,B} &= p_A \circ \alpha_{A,I,B} \\ &= r_A \circ p_{A \otimes I} \\ \\ p_B \circ (1_A \otimes l_B) \circ \alpha)_{A,I,B} &= l_B \circ p_{I \otimes B} \circ \alpha_{A,I,B} \\ &= p_B. \end{aligned}$$

5 The Pentagon Identity

Finally we must show that the associators satisfy the pentagon identity; i.e., for any three objects A, B and C in $\mathfrak{C}$, we require commutativity of the following diagram:

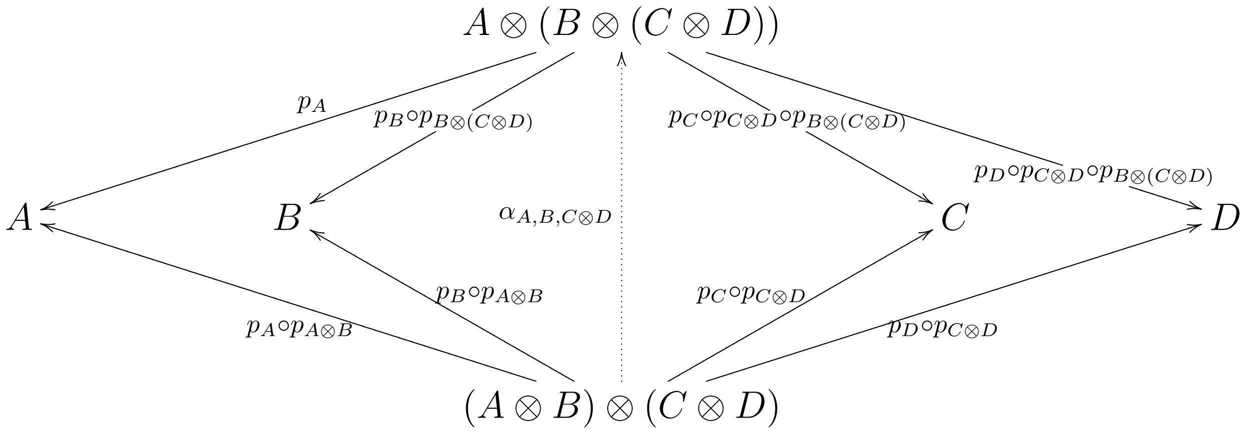


The proof of this proceeds as follows. All vertices of the pentagon are products of the objects A, B, C and D, and every edge is a map from one of these products to another. It suffices to show that each of these maps completes a commutative diagram required by the universal property of the product. Since these maps are unique, the result follows. We will show this for just one edge of the pentagon, the map

$$\alpha_{A,B,C \otimes D} : (A \otimes B) \otimes (C \otimes D) \to A \otimes (B \otimes (C \otimes D)).$$

The other edges are done in a similar fashion.

What we must show, then, is that this map provides the following commutative diagram:



For each vertex A, B, C and D, there are two routes to check. But the equality of each pair of routes follows directly in one step using the identities

$$\begin{aligned} p_A \circ p_{A\otimes B} &= p_A \circ \alpha_{A,B,C} \\ p_B \circ p_{A\otimes B} &= p_B \circ p_{B\otimes C} \circ \alpha_{A,B,C} \\ p_C &= p_C \circ p_{B\otimes C} \circ \alpha_{A,B,C} \end{aligned}$$

given in section 2.2.