COURSE NOTES ON QUANTIZATION AND COHOMOLOGY, FALL 2006

JOHN BAEZ
NOTES BY APOORVA KHARE
FIGURES BY CHRISTINE DANTAS

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA
RIVERSIDE, CA 92521, USA

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1. Preface

These are lecture notes taken at UC Riverside, in the Tuesday lectures of John Baez's Quantum Gravity Seminar, Fall 2006. The notes were taken by Apoorva Khare. Figures were prepared by Christine Dantas based on handwritten notes by Derek Wise. You can find the most up-to-date version of all this material here:

`http://math.ucr.edu/home/baez/qg-fall2006/`

Related notes on classical mechanics can be found here:

`http://math.ucr.edu/home/baez/classical/`

For the continuation of this seminar in Winter 2007, see:

`http://math.ucr.edu/home/baez/qg-winter2007/`

If you see typos or other problems with any of these notes, please let John Baez know (`baez@math.ucr.edu`).

2. Oct 3, 2006: Introduction

2.1. **Perspective.** Our aim in this course is to try and categorify "everything in the universe". More precisely, we want to replace sets by 1-categories, categories by 2-categories, and so on.

Analogy from physics:
(0) Initially, people studied particles (or static ones).

(1) Then, they went on to study the "legal paths" that such particles could take in the ambient space (legal according to the laws of physics). This led to *particle dynamics*, which extended the previous study of *particle statics.*

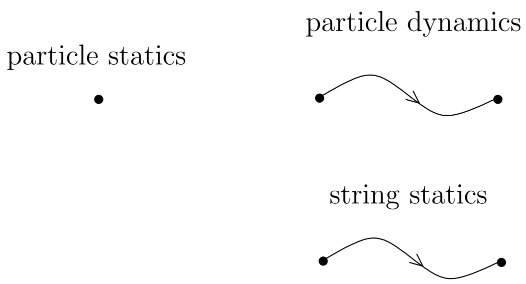


Figure 1.

(1′) Relatively recently, people have reformulated particles in motion as *strings*, which leads to calling particle dynamics as *string statics.* (Here, a string is merely a map from an interval into the ambient space.)

(2) This makes us want to consider *string dynamics* now.
(2′) But this should be the same as *2-brane statics.*

And so on...

In general, we can reinterpret p-brane dynamics as $p+1$-brane statics, for any $p \geq 0$.

Mathematics comes in: What happens in mathematical notation, is that we take a space X (a manifold, perhaps), and form
PX = path space of X, defined as $\{\gamma : [t_0, t_1] \to X : \gamma \text{ smooth}\}$,

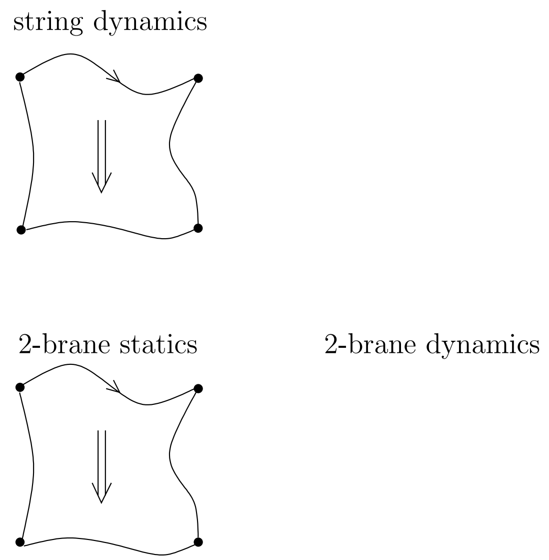


FIGURE 2.

PPX = path space of PX,
$\ldots$

As a variation, we can consider paths (*open strings*) vs. loops (*closed strings*). Thus, PX is the configuration space of open strings,; the analogous space for closed springs is the loop space LX. Thus, $LX = \{\gamma : S^1 \to X\}$, where elements are free loops (i.e. not based at a point).

Remark 2.1. Note that $LX \neq \{\gamma \in PX : \gamma(t_0) = \gamma(t_1)\}$ because these loops might have a "corner" at the basepoint, whereas LX was the space of smooth loops.

We can now similarly form $LX, LLX, \ldots$, and it is *these* spaces, that are related to the cohomology of X. The first cohomology group of a topological space X can be defined as $[X, U(1)] := \{\text{homotopy classes of maps} : X \to U(1)\}$. (Here, the unitary group $U(1) = S^1 \subset \mathbb{C}^*$ is the set of unit modulus complex numbers.) So, we're now asking how LX is related to $[X, U(1)] = [X, S^1]$.

2.2. **(Higher) cohomology and physics.** Let's see how maps $S : L^p(X) \to U(1)$ show up in the physics of closed p-branes, for various p. Note that by a particle, we'll simply mean (below) a point in a (configuration) space. For example, the position of n objects in X denotes a particle in the configuration space X^n (for any $n \in \mathbb{N}$).

Thus, we have a "configuration space" X, whose points $x \in X$ are possible positions for our particle (or the position of a general classical system).

We may decide that it is possible for two such particles to have the same position (or we may decide to ban such things) by considering $X \times X$ (or $X \times X \setminus \Delta_X$ respectively).

Now, often X is a manifold, and we choose a 1-form F on X, called the *force field*. Thus, $F : X \to T^*X$, or $F(x) \in T^*_x X$, or F is a section of T^*X (each formulation containing more information than the previous one).

Definition 2.2.

(1) The *work* done *on* the particle as it moves along a path $\gamma : [t_0, t_1] \to X$ is defined to be $W(\gamma) = \int_\gamma F \in \mathbb{R}$.
(2) A particle $x \in X$ is in *equilibrium* if $F(x) = 0$; that is, for all infinitesimal displacements $v \in T_x X$, we have $F(x)(v) = 0$. (In D'Alembert's terminology, the *virtual work* $F(x)(v)$ vanishes.)

Often, the 1-form F comes from the differential of a 0-form V, i.e. $F = -dV$ for some function $V : X \to \mathbb{R}$ called the *potential energy*. (Here, the negative sign is convention.)

Then at any critical point x of V, there is an equilibrium, meaning that $dV(x) = 0$. This is often (misleadingly?) called the *principle of least energy*, since often - but not always - x is a minimum.

The other advantage of having such an F, is that the Fundamental Theorem of Calculus (essentially) says that (recall the negative sign for F)

$$W(\gamma) = V(\gamma(t_0)) - V(\gamma(t_1))$$

2.3. **Classical dynamics vs. open string statics.** Morally, these are the same concept, except that we use PX instead of X, or more precisely, we use $P_{x_0 \to x_1} X := \{\gamma \in PX : \gamma(t_i) = x_i,\ i = 0, 1\}$ for some $x_0, x_1 \in X$.

The idea now is that a particle now chooses an "optimal" path to "go" from x_0 to x_1, i.e. $dS(\gamma) = 0$, where $S : P_{x_0 \to x_1} X \to \mathbb{R}$ is called the *action*. The equation $dS(\gamma) = 0$ is called the *principle of least action*.

The string picture: To think in terms of strings, we think of $P_{x_0 \to x_1} X$ as a configuration space of an (open) string, and S as a potential.

Remark 2.3. Note that the ends of the string are fixed here; to avoid this, we may move to a bigger space, namely PX. Or for closed strings, use LX. Or for based strings or loops, use $P_* X, L_* X$ etc.

For a fixed basepoint $* \in X$, the space $L_* X$ is also called $\Omega X := \{\gamma \in LX : \gamma(t_0) = *\}$.

We can now repeat this procedure by going to higher and higher (dimensional) branes. We'll thus need actions on these spaces, for example for $p = 2$, we need some function of the type

$$S : P_{\gamma_0 \to \gamma_1} P_{x_0 \to x_1} X \to \mathbb{R}$$

where all γ's start at x_0 and end at x_1, etc. Thus, we have

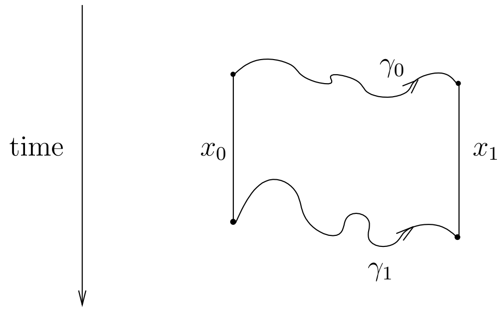


FIGURE 3. worldsheet Σ between two paths in $P_{x_0 \to x_1} X$

and we are trying to find the *worldsheet(s)* $\Sigma \in P_{\gamma_0 \to \gamma_1} P_{x_0 \to x_1} X$, so that $dS(\Sigma) = 0$. (Note that $P_{\gamma_0 \to \gamma_1} P_{x_0 \to x_1} X$ are just maps : $[0,1] \times [0,1] \to X$.)

D-branes: (The D stands for Dirichlet.) This means that we study branes with added boundary conditions. For insatnce, the Σ above fixes all initial points to be x_0, and all final points to be x_1. Thus, the boundaries are *coupled to 0-submanifolds* x_0, x_1.

In general, we can use submanifolds of dimension d, and look at p-branes coupled wth d-branes for boundary conditions.

2.4. **The quantum case.** The question: "How does the particle *know* in advance which path to take, before it has reached the end?", led to the study of the quantum versions of all these ideas. A possible explanation was first given by Richard Feynman, who said that the quantum dynamics of particles is also governed by $S : P_{x_0 \to x_1} X \to \mathbb{R}$, but in a new way:

Instead of choosing the path γ with $dS(\gamma) = 0$, it chooses *all* paths with certain "amplitudes", given by

$$e^{iS/\hbar} \in U(1)$$

where $\hbar$ is *Planck's constant*, in units of action. (We'll often choose units with $\hbar = 1$.) This is how $U(1)$ gets into the picture in physics - through quantization.

In retrospect, we can do classical dynamics of particles using not S, but $e^{iS} = A$, since $dA(\gamma) = 0$ makes sense (given $A : P_{x_0 \to x_1} X \to U(1)$, we can define a complex-valued form dA on $P_{x_0 \to x_1} X$), and moreover, $dA(\gamma) = 0 \Leftrightarrow dS(\gamma) = 0$.

Similarly, we can do classical statics of a particle, using not $V : X \to \mathbb{R}$, but $A = e^{iV} : X \to U(1)$.

So instead of using $dA = 0$ or integrating over a single path, we integrate over *all* paths, weighted by the $U(1)$-valued function In short, we have

$$\begin{aligned}
&\text{statics of particles} &&\longleftrightarrow A = e^{iV} : X \to U(1),\\
&\text{particle dynamics} = \text{statics of strings} &&\longleftrightarrow A = e^{iS} : PX \to U(1),\\
&\ldots,\\
&\text{statics of } p+1\text{-branes} &&\longleftrightarrow A : P^pX \to U(1).
\end{aligned}$$

If we restrict to $\Omega^p X \subset P^p X$, we get $A : \Omega^p X \to U(1)$, and in fact, $[\Omega^p X, U(1)] \cong H^{p+1}(X, \mathbb{Z})$! (Well, not exactly isomorphic, but close to it at any rate.)

This is how (higher) cohomology comes into the picture involving strings and higher branes!

3. Oct 10, 2006: Lagrangian Mechanics

Here's some homework to do, first of all. Work out the "statics of a spring in imaginary time"! The problem (and some notes on it) can be found at `http://www.math.ucr.edu/home/baez/classical/`.

Think of a rock thrown away from the earth. It chooses a parabolic path to come back to the ground, because this path minimizes the action.

A static (or hung) spring is traditionally of zero length because this minimizes energy.

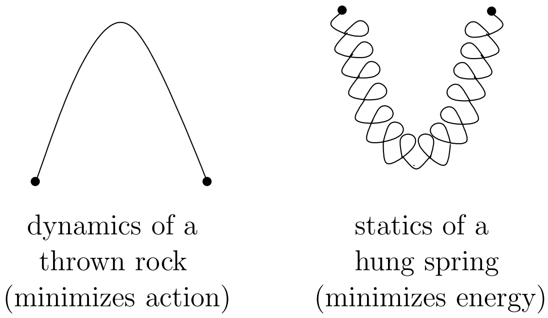


Figure 4. parabolic shapes - of rock trajectory and hung spring

In both case, the energy or motion or static state is affected by (or counters) gravity.

Moreover, the paths in these two cases are upside-down relative to one another because there is a sign change, which ultimately comes from $i^2 = -1$; hence, the notion of a spring in imaginary time!

To do this homework, one needs to know what we talk about today: Lagrangian mechanics.

3.1. Introduction to the Lagrangian approach.

3.1. Introduction to the Lagrangian approach. Suppose X is a (finite-dimensional) manifold, called the *configuration space*. We want a *law of physics* (which we will call the *Euler-Lagrange equation*) satisfied by paths $\gamma : [t_0, t_1] \to X$.

To get this, we define $P_{x_0 \to x_1} X := \{\gamma : [t_0, t_1] \to X,\ \gamma(t_i) = x_i\}$ as we did last time. This is a (smooth) infinite-dimensional manifold in its own right. (Do we also want it to be a Frechet manifold, i.e. locally homeomorphic to Frechet space?)

Moreover, we also choose a smooth function $S : P_{x_0 \to x_1} X \to \mathbb{R}$, called the *action*.

The Euler-Lagrange equation then (abstractly) says that

$$dS(\gamma) = 0$$

where $dS \in \Omega^1(P_{x_0 \to x_1}X)$. So, a particle follows a path that is the critical point of the action. Slightly more concretely, $dS(\gamma) \in T^*_\gamma(P_{x_0 \to x_1}X)$, and so we're saying that

$$dS(\gamma)(\delta\gamma) = 0 \ \forall \delta\gamma \in T_\gamma(P_{x_0 \to x_1}X)$$

(In physics notation, $\delta\gamma$ is called the "(infinitesimal) variation" in γ.)

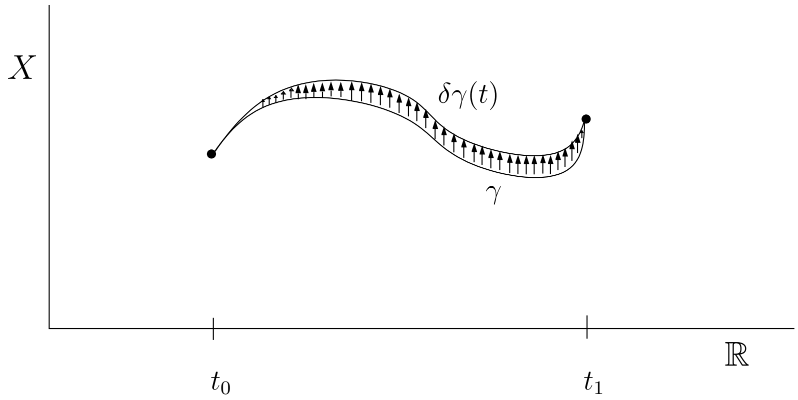


FIGURE 5. the plane as a t vs. X plot; how $\delta\gamma$ is a path

Thus, if we think of the *point* $\gamma \in P_{x_0 \to x_1}X$ as a path γ, then the tangent vector $\delta\gamma$ can be thought of as a path in the tangent bundle:

$$\delta\gamma(t) \in T_{\gamma(t)}X \subset TX$$

3.2. **Deriving the Euler-Lagrange equations.** In physics, we often have actions of the form

$$S(\gamma) = \int_{t_0}^{t_1} L(\gamma(t), \dot\gamma(t))\ dt$$

where $\gamma(t)$ stands for the position, $\dot\gamma(t)$ denotes the *velocity*, and L is the *Lagrangian*. Thus, $L : TX \to \mathbb{R}$ is a smooth function, where TX is the space of position-velocity pairs.

Remark 3.1. The path γ in X might go through several different coordinate charts/patches. However, we then break it up into small paths, each of which is in only one such chart. Thus, locally we work over a chart in X, so the situation is homeomorphic (diffeomorphic?) to working in $\mathbb{R}^n$, and the tangent bundle TX is then locally homeomorphic to $T\mathbb{R}^n \cong \mathbb{R}^n \oplus \mathbb{R}^n$.

Thus, our Lagrangian is written as $L(\{x_i\}, \{y_i\}) \in \mathbb{R}$, where $\{x_i\}$ are local coordinates for position vectors, and $\{y_i\}$ are local coordinates on the tangent space.

We now derive the Euler-Lagrange equations in the above situation. Given S as above, what does $dS(\gamma) = 0$ mean? It means that for all $\delta\gamma \in T_\gamma P_{x_0 \to x_1} X$, we have $0 = dS(\gamma)(\delta\gamma)$. We now expand this, and use Einstein summation notation henceforth.

$$0 = dS(\gamma)(\delta\gamma) = \int_{t_0}^{t_1} (\nabla L \cdot \delta\gamma)(t)\ dt = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x^i} \delta\gamma^i(t) + \frac{\partial L}{\partial y^i} \delta\dot{\gamma}^i(t) \right) dt$$

But $\delta\dot{\gamma}^i(t) = \frac{d}{dt}\delta\gamma^i(t)$. We now use integration by parts on the second term, to get

$$\begin{aligned} 0 = dS(\gamma)(\delta\gamma) &= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x^i} \delta\gamma^i(t) + \frac{\partial L}{\partial y^i} \frac{d}{dt} \delta\gamma^i(t) \right) dt \\ &= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x^i} \delta\gamma^i(t) - \left(\frac{d}{dt} \frac{\partial L}{\partial y^i} \right) \delta\gamma^i(t) \right) dt \end{aligned}$$

This works because the boundary terms in our integration vanish here, because the sum telescopes across coordinate charts - and at the “global” endpoints, (the picture shows that) $\delta\gamma(t_0) = \delta\gamma(t_1) = 0$.

Continuing with the calculations,

$$\int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial y^i} \right) \delta\gamma^i(t) = 0\ \forall \delta\gamma^i$$

But if we have smooth functions φ_i, so that $\int_{t_0}^{t_1} \varphi_i f^i = 0$ for all smooth functions f^i, then $\varphi_i \equiv 0$ for all i. Thus, the previous equation gives us the *Euler-Lagrange equations*:

$$\frac{\partial L}{\partial x^i} = \frac{d}{dt} \frac{\partial L}{\partial y^i}\ \forall i \tag{EL}$$

or, more pedantically,

$$\frac{\partial}{\partial x^i} L(\gamma(t), \dot{\gamma}(t)) = \frac{d}{dt} \frac{\partial}{\partial y^i} L(\gamma(t), \dot{\gamma}(t))\ \forall i$$

3.3. **Physics notation.** Physicists don’t write $\{x^i, y^i\}$ as coordinates on TX; they use $\{q^i, \dot{q}^i\}$, even though $\dot{q}^i$ here is not the time derivative of anything. They also write $q : [t_0, t_1] \to X$ instead of $\gamma : [t_0, t_1] \to X$, which makes the notation $\dot{q}^i$ ambiguous. (And no one cares!)

Therefore the physicists’ version of the Euler-Lagrange equations looks like

$$\frac{\partial L}{\partial q^i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i}$$

3.4. **Example: A particle in a potential.** Let $X = \mathbb{R}^n$ and consider $L : TX \to \mathbb{R}$ given by

$$L(q, \dot{q}) = \frac{1}{2}m||\dot{q}||^2 - V(q)$$

where $q \in X, \dot{q} \in T_qX$, and the terms on the right-hand side are the kinetic and potential energies. Thus, $m > 0$ is the mass, and $V : X \to \mathbb{R}$ is the *potential*.

This Lagrantian is "weird"; it's not the kinetic *plus* potential energies (which is the total energy), but rather, the kinetic energy (i.e. how much is happening) *minus* potential energy (how much could - potentially! - happen, but is not happening). This we call the total "happening-ness" $\ddot{\smile}$. (The reason, of course, is that the Lagrangian is *not* the energy; that's what comes from the Hamiltonian approach!)

Nature usually tries to minimize the integral of this over time.

Let us now carry out the computations. The Euler-Lagrange equations now say:

$$-\frac{\partial V}{\partial q^i} = \frac{\partial L}{\partial q^i} \stackrel{EL}{=} \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \frac{d}{dt}m\dot{q}_i = m\ddot{q}_i$$

or, in Newton's words, $F = ma$, where $F_i = -\frac{\partial V}{\partial q_i}$ is the *force*, and $a_i = \ddot{q}_i$ is the *acceleration*.

Definition 3.2. For any (smooth) Lagrangian function $L : TX \to \mathbb{R}$, we define the *momentum* to be $p_i := \frac{\partial L}{\partial \dot{q}^i}$, and the *force* to be $F_i := \frac{\partial L}{\partial q^i}$.

The Euler-Lagrange equations now say that $\dfrac{d}{dt}p_i = F_i$.

3.5. **"Sneak preview".** To relate this to cohomology, let's step back for a moment: we have derived classical mechanics from the principle of least action, based on $S : P_{x_0 \to x_1}X \to \mathbb{R}$, or more generally, $S : PX \to \mathbb{R}$.

It would be nice (at least, from the point of view of de Rham cohomology), if there were a 1-form $\alpha \in \Omega^1(X)$ so that $S(\gamma) = \int_\gamma \alpha$.

Also note that the action S in our example is not of this form (because paths can be reparameterized - whereby the integral above remains the same - but this action is not independent of the parametrization). For instance, a ball rolling along at constant speed, does not possess the same action function as a ball going one way, then reversing, and then going back again the "correct way" to the end.

In other words, $S(\gamma)$ depends here on the parametrization of γ. So, we need to write $S(\gamma)$ as the integral of some 1-form over a path (that we cook up from γ) in some *other* space.

Question. What is this other space?
Hint. For any manifold M, the cotangent bundle T^*M has a God-given 1-form on it, called the *canonical 1-form.*

Answer. We'll use this to get the job done, but with $M = X \times \mathbb{R}$. In other words, $M = X \times \mathbb{R}$ stands for *space-time*, so to speak, and we need the extra dimension to get the reparametrization-invariance.

This new manifold M is also known as the *extended configuration space*. Thus, we now reparametrize both space *and* time in terms of some other, arbitrary parametrization, and this helps us achieve $S = \int_\gamma \alpha$.

4. Oct 17, 2006: From Lagrangian to Hamiltonian Dynamics

4.1. **Recap.** We want a description of classical mechanics where the action is the integral of some 1-form along a path. This path will lie
not in X = configuration space $\ni$ position,
nor in $TX \ni$ (position, velocity),
nor in T^*X = phase space $\ni$ (position, momentum),
but in $T^*(X \times \mathbb{R})$ = extended phase space $\ni$ (position, momentum, time, energy).

(Thus, energy : time :: momentum : position.)

To get there, we first study the phase space T^*X and energy (also called the *Hamiltonian*). We start with a Lagrangian $L : TX \to \mathbb{R}$, and get the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i}$$

where $\frac{\partial L}{\partial \dot{q}^i} = p_i$ is the momentum, and $\frac{\partial L}{\partial q^i} = F_i$ is the force.

Notation: We use subscripts for cotangent vectors, and superscripts for tangent vectors.

4.2. **A matter of notation.** The first question we ask, is: Why is momentum a *cotangent* vector?

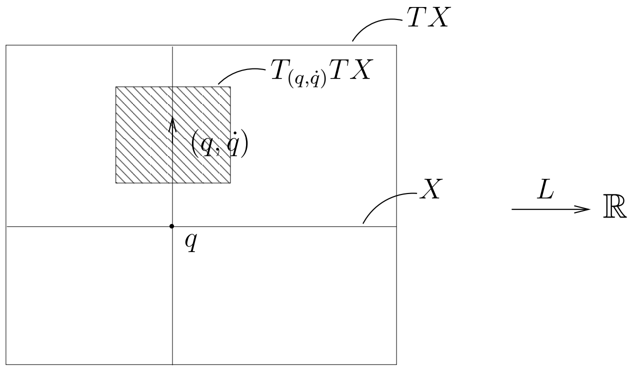


Figure 6. X vs. TX plane (e.g. $X = \mathbb{R}$)

Here, $\frac{\partial L}{\partial \dot{q}^i}$ describes the derivative of L in the vertical direction (i.e. along the fiber, or tangent space). In other words, {vertical vectors} $= T_{(q,\dot{q})}TX$.

Moreover, we have that the set of vertical vectors is precisely the kernel of the map $d\pi : T_{(q,\dot{q})}T_qX \to T_qX$ is the differential of the projection $\pi : TX \to X$ $((q,\dot{q}) \mapsto q)$.

But for any (real finite-dimensional) vector space V, we have $T_vV \cong V$ for all $v \in V$. So $T_{(q,\dot{q})}T_qX \cong T_qX$, whence momentum is really the derivative of $L : TX \to \mathbb{R}$, but only in the "vertical" directions.

Now, the derivative of L is the 1-form $dL(q, \dot{q}) : T_{(q,\dot{q})}TX \to \mathbb{R}$, and the vertical vectors in the domain are just T_qX. So the momentum is a linear map $p : T_qX \to \mathbb{R}$, i.e. $p \in T_q^*X$ is a cotangent vector.

4.3. **Switching to the Hamiltonian approach.** We will now switch from the "Lagrangian approach", based on $(q, \dot{q}) : TX \to \mathbb{R}$, to the "Hamiltonian approach", based on $(q, p) \in T^*X =$ *phase space*. We'll do this using the *Legendre transform* $\lambda : TM \to T^*M$, that takes $(q, \dot{q}) \mapsto (q, p)$, with $p_i = \frac{\partial L}{\partial \dot{q}^i}$. (Here, λ is defined using L.)

From now on, **assume** that L is *strongly regular*, i.e. $\lambda : TM \to T^*M$ is a diffeomorphism.

Example: We once again look at our familiar example of a particle moving on a Riemannian manifold (X, g), in a potential V. The Lagrangian is

$$L(q, \dot{q}) = \frac{m}{2} g(\dot{q}, \dot{q}) - V(q)$$

where $m > 0$ and $V : X \to \mathbb{R}$. (Thus, the components on the right-hand side are the kinetic and potential energies.) In what follows, not that $g(\dot{q}, \dot{q}) = g_{ij}\dot{q}^i\dot{q}^j$, whence $p_i = \frac{\partial L}{\partial \dot{q}^i} = mg_{ij}\dot{q}^j$. For this, we have used the coordinate-dependent form of the metric:

$$g(\dot{q}, -) = g_{ij}\dot{q}^j \in T_q^*X$$

So λ is a diffeomorphism, since g is nondegenerate (i.e. $\dot{q} \mapsto g(\dot{q}, -)$ is a bijection). In other words, L is strongly regular, as claimed. $\square$

4.4. **Energy.** To translate the Euler-Lagrange equations into equations satisfied by q, p, we need the concept of *energy*.

Theorem 4.1 (Conservation of Energy)**.** *Given any Lagrangian $L : TX \to \mathbb{R}$, and $q : [t_0, t_1] \to X$ satisfying the Euler-Lagrange equations, the function $E(q(t), \dot{q}(t))$ is independent of t, where $\dot{q}(t) = \frac{d}{dt}q(t)$, and $E : TX \to \mathbb{R}$ is given by $E(q, \dot{q}) = p_i\dot{q}^i - L(q, \dot{q}) = \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L(q, \dot{q})$.*

Proof. We compute:

$$\begin{aligned} \frac{d}{dt}E(q, \dot{q}) &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)\dot{q}^i + \frac{\partial L}{\partial \dot{q}^i}\frac{d}{dt}\dot{q}^i - \frac{\partial L}{\partial q^i}\frac{d}{dt}q^i - \frac{\partial L}{\partial \dot{q}^i}\frac{d}{dt}\dot{q}^i \\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)\dot{q}^i - \frac{\partial L}{\partial q^i}\frac{d}{dt}q^i \end{aligned}$$

since two terms cancel. But now, $\frac{d}{dt}q^i = \dot{q}^i$, and we now apply the Euler-Lagrange equations to make the entire expression vanish, as claimed. $\square$

Back to our example: For a particle on (X, g) as above, with $L(q, \dot{q}) = \frac{m}{2} g_{ij}\dot{q}^i\dot{q}^j - V(q)$, we have

$$E(q, \dot{q}) = p_i\dot{q}^i - L(q, \dot{q}) = mg_{ij}\dot{q}^j\dot{q}^i - \frac{m}{2} g_{ij}\dot{q}^i\dot{q}^j + V(q) = \frac{m}{2} g_{ij}\dot{q}^i\dot{q}^j + V(q)$$

So the energy is indeed the *sum* of the kinetic and potential energies, just as the Lagrangian was their difference.

4.5. **Hamilton's Equations.** Using our diffeomorphism $\lambda : TX \to T^*X$, we can define the *Hamiltonian* to be

$$H = E \circ \lambda^{-1} : T^*X \to \mathbb{R}$$

In other words, $H(q, p) = E(q, \dot{q})$, since $\lambda(q, \dot{q}) = (q, p)$. Let us now figure out *Hamilton's equations*, that describe the time evolution of $(q, p) = \lambda(q, \dot{q})$, given that $(q, \dot{q})$ satisfy the Euler-Lagrange equations. Here's how: compute dH in two different ways.

Method 1. $H : T^*X \to \mathbb{R}$, and T^*X has local coordinates (q^i, p_i) coming from local coordinates q^i on X. So we get

$$dH = \frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial p_i} dp_i$$

Method 2. But we can also compute dH using the coordinates $(q^i, \dot{q}^i)$. These are really local coordinates on TX coming from local coordinates q^i on X, but they *become* coordinates on the cotangent bundle, using $\lambda : TX \xrightarrow{\sim} T^*X$. Thus, we get

$$dH = d(p_i\dot{q}^i - L(q, \dot{q})) = (dp_i)\dot{q}^i + p_i d\dot{q}^i - \frac{\partial L}{\partial q^i} dq^i - \frac{\partial L}{\partial \dot{q}^i} d\dot{q}^i$$

The second and fourth terms cancel by definition of momentum, and we now compare with the expression from the previous method, equating the coefficients for $dp_i, d\dot{q}^i$. This gives us

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad -\frac{\partial H}{\partial q^i} = \frac{\partial L}{\partial q^i}$$

But the last term above equals $\frac{d}{dt} p_i$, by the Euler-Lagrange equations. Therefore, given a path q satisfying the Euler-Lagrange equations, we get *Hamilton's equations*:

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}$$

5. Oct 24, 2006: Hamiltonian Mechanics and Symplectic Geometry

5.1. **Recap.** We have seen that any Lagrangian $L : TX \to \mathbb{R}$ gives Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i}$$

describing a *flow* on TX, i.e. describing evolution (at least, such a global solution exists if L is well-behaved). We also get a Legendre transform $\lambda : TM \to T^*M$, and if L is strongly regular (i.e. λ is a diffeomorphism), then we get a flow on T^*M, describing the time evolution of position q and momentum p, which satisfies Hamilton's equations

$$\frac{d}{dt}q^i = \frac{\partial H}{\partial p_i}, \qquad \frac{d}{dt}p_i = -\frac{\partial H}{\partial q^i}$$

where $H : T^*M \to \mathbb{R}$ is the Hamiltonian, given by

$$H(q,p) = p_i\dot{q}^i - L(q,\dot{q}) = p_i\dot{q}^i - L(\lambda^{-1}(q,p))$$

Remark 5.1. Note that the Euler-Lagrange equations were "one" second-order equation, whereas the Hamiltonian equations are "two" first-order equations. This is the same, because there is a standard way to make an nth order equation into n first-order equations, by introducing auxiliary ("intermediate") variables.

Also recall our indexing notation: subscripts are used for cotangent vectors, and superscripts for tangent vectors.

5.2. **Some musical operators.** Suppose (X, g) is a Riemannian manifold, and $V : X \to \mathbb{R}$ is the "potential energy". Then the (nonrelativistic) Hamiltonian for a particle of mass $m > 0$ is the sum of the kinetic and potential energies, namely,

$$H(q,p) = \frac{|p|^2}{2m} + V(q)$$

Here, Hamilton's equations say

$$\frac{d}{dt}q^i = \frac{\partial H}{\partial p_i}, \qquad \dot{q}^i = \frac{p^i}{m}$$

Question. We only saw p_i's above. What is p^i?
Answer. $p^i = g_{ij}p_j$, where g is the Riemannian metric.
Reason. Force must be a cotangent vector, because the integral of force is work, which indicates that we are essentially integrating a 1-form on a path to get a number. But 1-forms come from the cotangent bundle.

Now, the time derivative of the momentum is force, so momentum should also be a cotangent vector! But momentum is related to velocity (i.e. d(position)), so this appears to be a tangent vector! What's wrong?

Solution. Absolutely nothing is wrong! However, this phenomenon (and the $p_i \leftrightarrow p^i$ issue) achieves consistency only when we introduce an isomorphism that we need, between T_qX and T_q^*X. This is the *lowering index operator* (that takes p^i to p_i), or the *flat* operator

$$\flat : T_qX \xrightarrow{\sim} T_q^*X \ \forall q \in X$$

And a Riemannian metric gives it to us! Since the metric is nondegenerate, hence we get an isomorphism $\flat$, given by

$$\flat(v) = g_q(v, -), \ v \in T_qX$$

We use this isomorphism - or more precisely, its inverse map, called the *sharp* operator $\sharp$ ("natural"ly (pun intended) ☺!). This turns the cotangent vector p into a tangent vector $\flat^{-1}(p) = \sharp(p) = p^\sharp$.

Thus, if g_q^{ij} is the inverse of $(g_{ij})_q = g_q$, then

$$p^\sharp = (p^i)_i = (g^{ij}p_j)_i$$

This equation, which had only to do with the kinetic energy, is "boring"; it is the other one (among Hamilton's equations), which deals only with the potential energy, that is interesting:

$$\frac{d}{dt}p_i = -\frac{\partial H}{\partial q^i} \Rightarrow \dot{p}_i = -\frac{\partial V}{\partial q^i} \Rightarrow ma = F$$

5.3. **The Hamiltonian vector field.** Let us now seek a coordinate-free formulation of Hamilton's equations. These give a vector field v_H on T^*X, describing how $(q(t), p(t))$ moves around. v_H is called the *Hamiltonian vector field*:

$$\frac{d}{dt}(q(t), p(t)) = v_H(q(t), p(t)) \in T(T^*X)$$

so $M = T^*X$ always. Moreover, $v_H = d(q, p)$, so

$$v_H = \frac{dq^i}{dt}\frac{\partial}{\partial q^i} + \frac{dp_i}{dt}\frac{\partial}{\partial p_i}$$

Here, $\frac{\partial}{\partial q^i}$ and $\frac{\partial}{\partial p_i}$ are a basis of vector fields on some open set in T^*X, coming from local coordinates (q^i, p_i) on T^*X.

But now, Hamilton's equations allow us to rewrite v_H as

$$v_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}$$

Example: Say $X = \mathbb{R}^n$, so $T^*X \cong \mathbb{R}^{2n}$. Then, as the picture below shows, if we identify these cotangent vector fields dH, v_H with tangent vectors using the metric, then they are orthogonal at each point. This is because

$$dH = \left(\frac{\partial H}{\partial q^i}, \frac{\partial H}{\partial p_i}\right), \qquad v_H = \left(\frac{\partial H}{\partial p_i}, -\frac{\partial H}{\partial q^i}\right)$$

(so one can easily verify that their dot product vanishes). Moreover, the gradient dH is always normal to the level curves, so (at least if $n = 1$) we get that v_H must lie along the vector field!

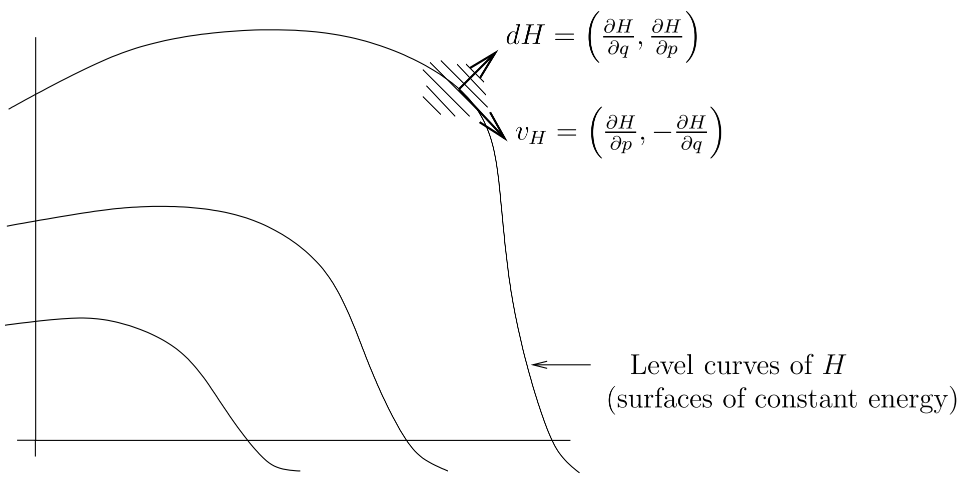


FIGURE 7. level curves for H, and how $dH \perp v_H$ on a p-q plot

Furthermore, we can recover v_H from the flow, and H from v_H (upto a scalar), on any connected component of the manifold.

Back to our example, in the graph above, dH is really a 1-form, but in $\mathbb{R}^2$, we can use the metric to turn it into a vector field ∇H, and rotate this $90°$ clockwise to get v_H. So, any solution of Hamilton's equations moves along the level curves of H, and conservation of energy follows automatically!

But back to our original motivation: to seek a coordinate-free description of Hamilton's equations. Recall from above, that one uses Hamilton's equations to rewrite v_H. Thus, we seek a coordinate-free description of how to turn the 1-form dH into the vector field v_H.

One way to turn 1-forms into vector fields is to use the Riemannian metric: use $\flat, \sharp$. But, $\sharp(dH) = \nabla H$ is not parallel (proportional), but perpendicular (normal) to the level curves of H!

So, instead of a metric, we would need to use an *anti-symmetric* nondegenerate bilinear form ω on $M = T^*X$.

The good news is that every cotangent bundle $M = T^*X$ is automatically equipped with such an $\omega : TM \times TM \to \mathbb{R}$. This is called a *symplectic* form.

5.4. **Homework.** In our example, we have $\omega = dq^i \wedge dp_i$ (summing over i, of course). Show that with this 2-form on $M = T^*X$, we have $\omega(v_H, -) = dH$.

Proof. This is an exercise in wedge calculations. As a warmup, we note some random facts about wedges. For instance, $f \wedge dx = f\ dx$, and

$$\begin{aligned} dx \wedge dy \wedge \ldots (f\frac{\partial}{\partial x}, -) &= f\ dy \wedge \ldots \\ dx \wedge dy \wedge \ldots (f\frac{\partial}{\partial y}, -) &= -f\ dx \wedge \ldots \\ dp_i(\frac{\partial}{\partial q^j}) = dq^j(\frac{\partial}{\partial p_i}) &= 0\ \forall i, j \\ dp_i(\frac{\partial}{\partial p_j}) = dq^j(\frac{\partial}{\partial q^i}) &= \delta_i^j\ \forall i, j \end{aligned}$$

where δ_i^j is the Kronecker delta. We now show the result. Given $H : T^*X = M \to \mathbb{R}$, we have

$$v_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} \in TM$$

We also have $dH \in T^*M$, and $\omega = dq^k \wedge dp_k$. We now compute, using the "warmup facts" above:

$$\begin{aligned} \omega(v_H, -) &= \frac{\partial H}{\partial p_i}\delta_i^k dp_k - dq^k \wedge \frac{\partial H}{\partial q^i} dp_k(\frac{\partial}{\partial p_i}) = \frac{\partial H}{\partial p_i} dp_i + \delta_k^i \frac{\partial H}{\partial q^i} dq^k \\ &= \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q^i} dq^i = dH \end{aligned}$$

□

Another way of stating this result, is to *define* $\flat : TM \to T^*M$, given by $v \mapsto \omega(v, -)$. We once again say that ω is nondegenerate if $\flat$ is an isomorphism. If this happens, then the inverse $\sharp$ of $\flat$ takes T^*M to TM, and we can set $v_H = \sharp(dH)$ now.

5.5. **Coordinate-free formulations.** Getting back to our original problem: we wanted to write Hamilton's equations in a coordinate-free way. To do this, we needed to write v_H in a coordinate-free way. By the above homework problem, we only need to write ω in a coordinate-free way! And we do this now.

This is, moreover, good from another point of view: the definition of ω then stays the same under *more* symmetries - not just only those that fix a particular choice of coordinates.

So, how do we write this? In fact, $\omega = -d\alpha$, where $\alpha = p_i\ dq^i$ is a 1-form on T^*X, called the *canonical 1-form*, which can be defined without using coordinates! (We show this next time.) First check this:

$$d\alpha = d(p_i\ dq^i) = dp_i \wedge dq^i + p_i(-1)^0 d(dq^i) = -\omega$$

since p_i is a 0-form for all i.

Now, how do we define α without using coordinates? Keep in mind that α likes to eat tangent vectors to $M = T^*X$, e.g. $v \in T_{(q,p)}(T_q^*X)$, and spit out numbers.

Well, we have the projection π_1 from T^*X, taking (q,p) to q. Differentiate π_1 to get $d\pi_1 : T(T^*X) \to TX$. Thus, we can form $d\pi_1(v) \in T_qX$ (which is the analogue of dq^i). But we also have $p \in T_q^*X$, which is the second (component of the first) projection $\pi_2' : T(T^*X) \to T^*X$. So we get

$$\alpha(v) = p_i \; dq^i(v) = p(d\pi_1(v)) \in \mathbb{R}$$

Conclusion: Therefore, in coordinate-free terms,

$$\alpha(v) = \pi_2'(v)(d\pi_1(v)), \qquad \omega = -d\alpha$$

Next time we will see how this agrees, given a choice of coordinates p_i, q^i, with the usual formula $\alpha = p_i \; dq^i$ (note that some people write this as $p_i \wedge dq^i$ too).

6. Oct 31, 2006: More on the canonical 1-form

Let us draw a picture to illustrate the coordinate-free definition of α from last class. Note that in it, $q \in X, p \in T_q^*X, v \in T_{(q,p)}T_q^*X$, and $d\pi_1(v)$ is the "shadow" or the projection, of v.

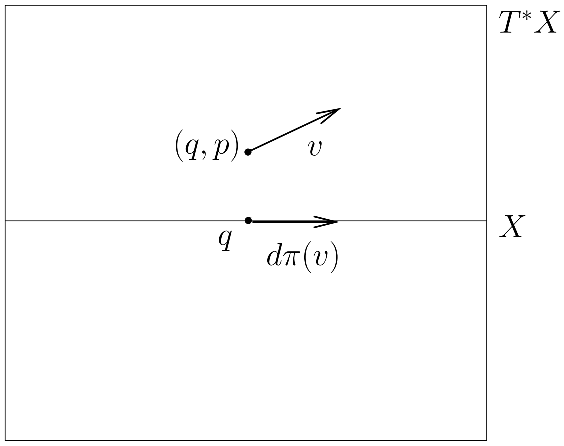


Figure 8. X vs. T_q^*X plot, to show how $d\pi_1(v)$ is the "shadow"

This picture doesn't make it clear, though, that the vertical fibers are *dual* to tangent vectors.

6.1. Reconciling with the coordinate-based definition.

Theorem 6.1. *If q^i are any local coordinates on X, and (q^i, p_i) the corresponding local coordinates on T^*X, then $\alpha = p_i\ dq^i$.*

Proof. Choose any $v \in T_{(q,p)}T_q^*X$. Then

$$v = \sum_i a^i \frac{\partial}{\partial q^i} + b_i \frac{\partial}{\partial p_i}$$

for some choice of scalars a^i, b_i, say. Evaluating at $p_i\ dq^i$, we have

$$p_i\ dq^i(v) = p_i\ dq^i(a^k \frac{\partial}{\partial q^k} + b_k \frac{\partial}{\partial p_k})$$

Since dq^i kills all $\frac{\partial}{\partial p_k}$ and almost all $\frac{\partial}{\partial q_k}$'s too, we just end up with $a^i p_i$.

On the other hand, $\alpha(v) = \pi_2'(v)(d\pi_1(v)) = p(a^j \frac{\partial}{\partial q^j})$ because $d\pi_1$ removes the vertical components $\frac{\partial}{\partial p_j}$. Now, recall that in α, the q^i is already a coordinate on $M = T^*X$. But the dq^i's form coordinates on T^*X because

q^i's are coordinates on X itself! So there's some sort of ambiguity here, but this is resolved if we remember that $d\pi_1$ takes one of these q's to the other.

In other words, in coordinates, it just throws away the vertical part, so we really are fine, in using q^i's in both instances.

This means that $p = p_i\ dq^i$, whence $\alpha(v) = p_i\ dq^i(a^j \frac{\partial}{\partial q^j}) = a^i p_i$ once again.

Thus, both definitions agree on every v, whence they are equal. □

6.2. **Symplectic manifolds.** There are various related ways in which α shows up in classical mechanics. We've now seen that $d\alpha = dp_i \wedge dq^i = -\omega$, where ω is a symplectic structure on $M = T^*X$.

Note that in general, more than just cotangent spaces have symplectic structures; these are precisely what allow us to treat M as a "phase space" (space of states of our system). That is, we have (position, momentum). Such M's are called *symplectic manifolds*, and it essentially means that we can do classical mechanics on them!

Definition 6.2. A *symplectic structure* on a (finite-dimensional) manifold M (which is then called a *symplectic manifold*) is a 2-form ω on M, such that

(1) ω is closed (i.e. $d\omega = 0$).
(2) ω is nondegenerate (i.e. the map $\flat : T_m M \to T^*_m M$, given by $v \mapsto \omega(v, -)$, is one-to-one - and since M is finite-dimensional, onto as well).

Note that this ω is skew-symmetric.

Remark 6.3. Every $M = T^*X$ has such a form on it. In fact, if $\omega = -d\alpha$ as earlier on $M = T^*X$, then we claim that ω is one such. For

$$d\omega = d(dq^i \wedge dp_i) = 0$$

by the Leibnitz rule, so that ω is closed. Moreover, to see that ω is nondegenerate, we need to show that $\flat$ is one-one. So suppose that $\flat$ kills $a^i \frac{\partial}{\partial q^i} + b_i \frac{\partial}{\partial p_i}$. Using our homework problem and the warmup random facts mentioned therein, we evaluate both sides at $\overline{a^i} \frac{\partial}{\partial p_i} - \overline{b_i} \frac{\partial}{\partial q^i}$ (write $a^i, b_i \in \mathbb{C}$). This gives us that

$$\sum_i |a^i|^2 + \sum_i |b_i|^2 = 0$$

so that $a^i = b_i = 0\ \forall i$, as required. □

How does ω on M allow us to do classical (Hamiltonian) mechanics?

As we said above, roughly speaking, the nondegeneracy of ω lets us define a *Hamiltonian vector field* v_H (given $H : M \to \mathbb{R}$) on M, by: $v_H = \sharp(dH) = \flat^{-1}(dH)$.

This now lets us write Hamilton's equations, describing the time evolution of states.

Definition 6.4. Given $x(t) \in M$ for $t \in [t_0, t_1]$, we say that it satisfies *Hamilton's equations*, if $\frac{dx(t)}{dt} = v_H(x(t))$.

What does the closedness of ω do for us? Well, under "mild assumptions", we get a flow on M.

Definition 6.5. A *flow* on M is a smooth(?) map $F : \mathbb{R} \times M \to M$, denoted by $(t, x) \mapsto F_t(x) \in M$, so that

(1) $F_t F_s = F_{t+s}$ for all $t, s \in \mathbb{R}$.
(2) F_0 is the identity map.

In particular, $F_{-t} = F_t^{-1}$, whence F is a group homomorphism from $\mathbb{R}$ to the diffeomorphism group of M.

How do we get this flow? It describes time evolution, and satisfies Hamilton's equations:

$$\frac{d}{dt} F_t(x) = v_H(F_t(x))$$

So: use this equation to define the flow as the integral curve (solution) to a differential equation. Then the closedness of ω can be used to show that: Given any H, the corresponding flow F_t preserves ω.

Formalism: How does F_t act on ω? In general, given a "good" (or smooth) $\varphi : M \to N$, we get $d\varphi : TM \to TN$, and one can now define the *pullback* map $\varphi^* : \Omega^p(N) \to \Omega^p(M)$ for any $p \geq 0$, by

$$(\varphi^* \omega)_m(v_1, \ldots, v_p) := \omega_{\varphi(m)}(d\varphi(v_1), \ldots, d\varphi(v_n))$$

where $v_i \in T_m M$ and $d\varphi(v_i) \in T_{\varphi(m)} N$ for all i.

6.3. **Digression on five-body systems.** The "mild assumptions" that allow us to derive a flow on M, rule out some physics "real-life" situations. For instance, the *n-body problem*: describe the time evolution of a system of n particles that interact gravitationally. For $n = 5$, it has been shown that they split off into two pairs and a fifth

[figure: two pairs orbiting each other, fifth particle orbits around]

where the paired objects "revolve" around each other, and the fifth particle orbits around both systems. As time passes, the distance $d(t)$ between the two paired systems increases, and the energy (Hamiltonian) involves $1/d(t)^2$, so the mild assumptions don't work here! Moreover, $d(t) \to \infty$ as t goes to some finite time point, and as $d(t)$ increases, the potential energy decreases, so the kinetic energy increases. That is, the fifth particle travels faster on its (larger and larger) orbit. Eventually, in finite time, its speed becomes infinite!

The problem is that using point particles allows us to get infinity as a limit in finite time. For instance, if both particles were points, they could "superimpose", resulting in the realisation of an infinite amount of potential energy.

In reality, such a scenario only occurs for black holes. Thus, there are problems in such situations, which is why we need the theory of relativity etc.

6.4. **The 1-form and action.** There is another (related) way in which $\alpha \in \Omega^1(T^*X)$ shows up in physics. We can use it to describe the action of a path $q : [t_0, t_1] \to X$, our configuration space, as follows: We get $(q, \dot{q}) : [t_0, t_1] \to TX$, whence $\gamma : [t_0, t_1] \to T^*X$, our phase space. This is given by $t \mapsto \lambda(q(t), \dot{q}(t)) = (q(t), p(t))$, if we choose a Lagrangian $L : TX \to \mathbb{R}$, and use it to define the Legendre transform $\lambda : TX \to T^*X$.

In this case, set $H = p_i\dot{q}^i - L, \alpha = p_i \; dq^i$. Then the action of our path q is

$$S(q) = \int_{t_0}^{t_1} L(q, \dot{q}) \; dt = \int_{t_0}^{t_1} p_i(t)\dot{q}^i(t) \; dt - \int_{t_0}^{t_1} H(q(t), p(t)) \; dt$$

But the first term is the integral of $p_i(t)\frac{dq^i(t)}{dt}dt = \alpha$, so we conclude that

$$S(q) = \int_\gamma \alpha - \int_{t_0}^{t_1} H(q(t), p(t)) \; dt$$

(Thus, the integral along a path is "almost" the action of that path; so, we're amost doing Lagrangian mechanics.) In particular, we see that the *Principle of Least Action*

$$\delta(S(\gamma)) = 0$$

is equivalent to the principle $\delta(\int_\gamma \alpha) = 0$, as long as we let γ vary only over the paths *which conserve energy*! (Because we would need $\int_{t_0}^{t_1} H(q, p)$ to be just a number.) Therefore we only consider the paths in the set

$$M_E^1 := \{\gamma : [t_0, t_1] \to T^*X : q(t_i) = q_i, H(q(t), p(t)) = E \; \forall t\}$$

and then we would get the action to be

$$S(\gamma) = -(t_1 - t_0)E + \int_\gamma \alpha \; \forall \gamma \in M_E^1$$

7. Nov 07, 2006: The Extended Phase Space

7.1. **Aside.** As we saw last time, the canonical 1-form α on a symplectic manifold $M = T^*X$ occurs in classical mechanics in two ways:

(1) We can write down Hamilton's equations.
(2) We can look at the action along a path, so long as we only consider paths γ so that $H \equiv E \in \mathbb{R}$ on γ.

(In this second case, $S(\gamma) = \int_\gamma \alpha - E(t_1 - t_0)$, as we saw last time.)

Putting these together, we can give a physical interpretation of $-\int_\Sigma \omega$, where Σ is a surface with boundary γ for some loop γ. *Stokes' Theorem* says that

$$-\int_\Sigma \omega = \int_\Sigma d\alpha = \int_{\partial\Sigma} \alpha = \int_\gamma \alpha = S(\gamma) + \int_{t_0}^{t_1} H\ dt$$

which is just $S(\gamma)$ in the limit $t_1 \to t_0$, where this amounts to reparametrizing γ suitably (i.e. covering it in smaller and smaller time).

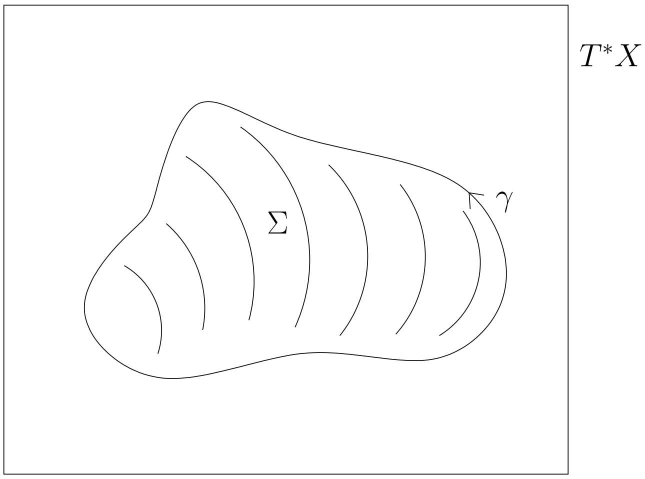


Figure 9. γ bounds Σ

Thus, the job of ω is to tell us how much action it costs, to run around the surface Σ, by:

$$\omega \mapsto -\lim_{t_1 \to t_0} \int_\Sigma \omega = S(\partial\Sigma)$$

7.2. **Bringing in spacetime.** Let us now consider the *extended phase space* $T^*(X \times \mathbb{R})$. Thus, our configuration space X gets replaced by *space-time*, or the *extended* configuration space $X \times \mathbb{R}$, and a point (q, t) in it says precisely *where* and *when* the system is. In the extended phase space, we thus add two coordinates from the phase space: time and energy.

The notation used is $(q, t, p, p_0) \in T^*(X \times \mathbb{R})$, with the terms denoting (position, time, momentum, negative energy(!)) respectively. Thus, the energy is given by $E = -p_0$.

In these terms, any path $\gamma : [t_0, t_1] \to T^*X$ gives rise to $\widetilde{\gamma} : [t_0, t_1] \to T^*(X \times \mathbb{R})$, given by

$$\widetilde{\gamma} : t \mapsto (q(t), t, p(t), -H(q(t), p(t)))$$

so note that we *still* do need a Hamiltonian $H : T^*X \to \mathbb{R}$.

The other thing to note, is that later on, we will replace time by any old parameter s, so that time would be a function $t(s)$. This would make the entire procedure independent of parametrization!

Now, the action is

$$S(\gamma) = \int_{t_0}^{t_1} p_i \dot{q}^i - H(q,p)\ dt = \int_{t_0}^{t_1} p_i\ dq^i - \int_{t_0}^{t_1} H(q,p)\ dt = \int_{\widetilde{\gamma}} \widetilde{\alpha}$$

where (comparing with the original 1-form α) $\widetilde{\alpha}$ is the canonical 1-form on $T^*(X \times \mathbb{R})$:

$$\widetilde{\alpha} = \alpha + p_0\ dt = p_i\ dq^i + p_0\ dt$$

(so we will use the notation $t = q^0$!) because we had cleverly chosen the p_0-coordinate of the path $\widetilde{\gamma}$ to be $-H(q(t), p(t))$.

(In other words, $\widetilde{\alpha}$ is canonical, but H is not. However, the "non-canonical-ness" is built into the choice of $\widetilde{\gamma}$, and then $\widetilde{\alpha}$ just gets $\widetilde{\gamma}$ into $S(\gamma)$!)

7.3. **Hamilton's equations and the conservation of energy.** Carlo Rovelli (see link on webpage!) has reformulated this, to apply not just to particles, but strings (and higher-dimensional "branes"). This requires a *canonical 2-form* (or higher forms).

To do this, let us allow $\widetilde{\gamma}$ to be more general: $\widetilde{\gamma} : [s_0, s_1] \to T^*(X \times \mathbb{R})$ is given by

$$s \mapsto (q(s), t(s), p(s), -H(q(s), p(s)))$$

where $H : T^*X \to \mathbb{R}$ is as before. In other words, $\widetilde{\gamma} : [s_0, s_1] \to Y \subset T^*(X \times \mathbb{R})$, where Y "contains all information about H":

$$Y = \{(q, t, p, p_0) : p_0 = -H(q, p)\}$$

This is a codimension-1-submanifold, hence is odd-dimensional, hence not a symplectic manifold. But it has a 1-form on it, namely, $\widetilde{\alpha}|_Y = i^*\widetilde{\alpha}$, the pullback of the inclusion $i : Y \hookrightarrow T^*(X \times \mathbb{R})$. Similarly, we also have a 2-form on Y, namely, $\widetilde{\omega}|_Y = i^*\widetilde{\omega}$.

Aside: This kind of situation is called a *contact manifold*, where this is an odd-dimensional manifold, with a foliation whose leaves are symplectic (codimension 1) submanifolds. Here, we foliate along the time coordinate...

Why is this approach nice? For one thing, we have

Proposition 7.1. *Assume* $\frac{dt}{ds} \neq 0$. *Then a curve* $\widetilde{\gamma} : [s_0, s_1] \to Y$ *gives a solution to Hamilton's equations, if and only if its tangent vector* $\widetilde{\gamma}'(s)$ *satisfies* $\widetilde{\omega}|_Y(\widetilde{\gamma}'(s), -) \equiv 0$. *Moreover, if this holds, then conservation of energy automatically holds.*

Remark 7.2.

(1) This is analogous to $\omega(v_H, -) = dH$ in the Hamiltonian approach above. But $\widetilde{\omega}$ is nondegenerate, so should it imply that $\widetilde{\gamma}'(s) = 0$? No, because $\widetilde{\omega}|_Y$ may well be degenerate!

(2) However, it does imply that some (most!) things about $\widetilde{\gamma}'(s)$ must vanish.

Proof. We compute:

$$\widetilde{\gamma}'(s) = \frac{dq^i}{ds}\frac{\partial}{\partial q^i} + \frac{dp_i}{ds}\frac{\partial}{\partial p_i} + \frac{dt}{ds}\frac{\partial}{\partial t} + \frac{dp_0}{ds}\frac{\partial}{\partial p_0}$$

But as we are in Y, hence

$$\frac{dp_0}{ds} = -\frac{\partial H}{\partial q^i}\frac{dq^i}{ds} - \frac{\partial H}{\partial p_i}\frac{dp_i}{ds}$$

Moreover, $\widetilde{\omega} = dp_i \wedge dq^i + dp_0 \wedge dt$, so wedge computations give

$$\widetilde{\omega}|_Y(\widetilde{\gamma}'(s), -) = -\frac{dq^i}{ds}dp_i + \frac{dp_i}{ds}dq^i - \frac{dt}{ds}dp_0 - \left(\frac{\partial H}{\partial q^i}\frac{dq^i}{ds} + \frac{\partial H}{\partial p_i}\frac{dp_i}{ds}\right) dt$$

Now, note that dp_0 and dt are linearly independent on $T^*(X \times \mathbb{R})$, but they are dependent on Y! In fact, on Y, we have

$$\frac{dt}{ds}dp_0 = -\frac{dt}{ds}dH = -\frac{dt}{ds}\left(\frac{\partial H}{\partial q^i}dq^i + \frac{\partial H}{\partial p_i}dp_i\right)$$

We now put this back into the computation above, and collect coefficients. Thus, for $\widetilde{\omega}(\widetilde{\gamma}'(s), -)$ to vanish on Y, is equivalent to the dq^i, dp_i, dt-parts all vanishing. In other words, if and only if

$$\begin{aligned} \frac{dp_i}{ds} &= \frac{dt}{ds}\frac{\partial H}{\partial q^i} \\ \frac{dq^i}{ds} &= -\frac{dt}{ds}\frac{\partial H}{\partial p_i} \\ \frac{\partial H}{\partial q^i}\frac{dq^i}{ds} + \frac{\partial H}{\partial p_i}\frac{dp_i}{ds} &= 0 \end{aligned}$$

Note that the left-hand side in the last equation is just $\dfrac{dH}{ds}$.

How do we get Hamilton's equations (and conservation of energy) out of this? Well, if t is a locally invertible function of s, then the equations above can be converted, using the Chain Rule, to involve $\frac{df}{dt}$ instead of $\frac{df}{ds}$. This gives us an equivalence with the new (modified) equations

$$\begin{aligned} \frac{dp_i}{dt} &= \frac{\partial H}{\partial q^i} \\ \frac{dq^i}{dt} &= -\frac{\partial H}{\partial p_i} \\ \frac{dH}{dt} &= \frac{dH}{ds} / \frac{dt}{ds} = 0 \end{aligned}$$

In other words, $\widetilde{\omega}|_Y(\widetilde{\gamma}'(s), -)$ is identically zero if and only if Hamilton's equations and conservation of energy hold.

The proof will be complete if we can show that the first two equations automatically imply the third. But this is obvious - we compute, using Hamilton's equations (or the equations in terms of s instead of t, as above):

$$\frac{dH}{ds} = \frac{\partial H}{\partial q^i}\frac{dq^i}{ds} + \frac{\partial H}{\partial p_i}\frac{dp_i}{ds} = \frac{dt}{ds} \cdot \left(\frac{\partial H}{\partial q^i} \cdot -\frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial p_i} \cdot \frac{\partial H}{\partial q^i} \right) = 0$$

□

Remark 7.3. As mentioned above, $\widetilde{\omega}|_Y(\widetilde{\gamma}'(s), -) \equiv 0$ implies that "most components" of it vanish. In other words, there are a lot of constraints. More precisely, we can now say what constraints there are. They are exactly the Hamilton equations above.

7.4. **Digression of the day: LIGO.** Neutron stars and black holes colliding among themselves (or each other) are the only known cosmic events that radiate enough gravitational energy to be detectable on Earth. There is a project funded by NASA (with the largest funding, barring the expedition to send men to Mars!) which attempts to measure such effects. It is called LIGO.

However, the effects are so small compared to local gravitational effects (like people walking, even!) that the instruments (detectors) have to be extremely sensitive to measure this. Thus, there are L-shaped mirrors in rooms only four degrees above absolute zero; the mirrors themselves hang from sapphire wires, and so on. Attempts are being made to get everything as precise and sensitive as possible, but instead of the 10^{-18} order of magnitude, we are a few orders away from this.

7.5. **A look back at the special case $t(s) = s$.** Let us keep the notation and the setup as above - where we work with the extended phase space $T^*(X \times \mathbb{R})$. Earlier, we considered $\gamma : [t_0, t_1] \to X$, giving rise to the phase space (points have coordinates of position and momentum here), together with the extra-special parametrization of the time function: $t(s) = q^0(s) = s$. We re-view the old theory now, from the point of view of the new one.

In general, the conservation of energy result in Lecture 3 holds in the extended phase space too, which means that the summation is starting from $i = 0$ (in q^i). Moreover, we keep our old Lagrangian $L(q, \dot{q})$, which is *independent of* p_0, q^0.

Since $q^0 = t$, hence $\dot{q}^0 = 1$, and the (extended) Hamiltonian is given by

$$H_{new}(q,p) = p_0 \cdot 1 + p_i \dot{q}^i - L(q,\dot{q}) = p_0 + H_{old}(q,p)$$

Thus, classical mechanics (what we saw in earlier classes) comes under the case $H_{new} = 0$! Because once we have this, we get that p_0 is independent of time, since conservation of energy tells us that $H_{old}(q,p)$ was. Moreover, $p_0 = -E$, where $E = H_{old}(q,p)$.

Moreover, $\widetilde{\alpha} = \alpha + p_0 dq^0 = \alpha + p_0 \; dt$, and p_0 is independent of time. Since $t(s) = s$, hence we are integrating on $[t_0, t_1]$, and we get

$$S(\widetilde{\gamma}) = \int_{\widetilde{\gamma}} \widetilde{\alpha} = \int_\gamma \alpha + p_0 \int_{t_0}^{t_1} dt = \int_\gamma \alpha + p_0(t_1 - t_0) = \int_\gamma \alpha - (t_1 - t_0)E = S(\gamma)$$

or, in other words, the old notion of action really *is* the integral of a 1-form on our path. We will see, in later classes, how this generalizes to higher-dimensional membranes.

8. Nov 14, 2006: From particles to strings and higher membranes

Let us now generalize everything we did so far for point particles

$$\gamma : [s_0, s_1] \to X \times \mathbb{R} = M$$

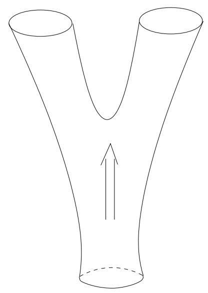
(where M is "space-time") to strings and higher membranes. Thus, we replace the one-dimensional $[s_0, s_1]$ by Σ, where now Σ is a p-dimensional manifold with boundary (or with corners, like a p-(hyper)cube $[0,1]^p$).

Moreover, we write $\gamma : \Sigma \to M$ (and also start to forget that we ever had the decomposition $M = X \times \mathbb{R}$ in an explicit way! This is what we mean when we talk about the "indistinguishability of the space and time directions"). The image of γ is now a p-dimensional membrane, which string theorists unfortunately call a '$(p-1)$-brane', since they're thinking about the 'spatial' dimensions of Σ instead. For example, in the case $p = 2$, we have a string, which they would call a 1-brane because it looks 1-dimensional at any moment of time, even though its 'worldsheet' (the image of γ) is 2-dimensional.

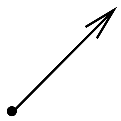
In the particle case, the canonical 1-form $\widetilde{\alpha}$ on T^*M played a key role in defining the action, Hamiltonian etc. We had defined $S(\widetilde{\gamma}) = \int_{\widetilde{\gamma}} \widetilde{\alpha}$, where $\widetilde{\gamma}$ is a path in T^*M. More precisely,

$$\widetilde{\gamma} : [s_0, s_1] \to Y \subset T^*M$$

where $Y = \{p_0 = -H(q,p)\}$ for our Hamiltonian $H : T^*X \to \mathbb{R}$. We also need a 'canonical p-form', and moreover, on what? (So that we can integrate it over our p-dimensional membrane.) We make the following table:

Particles ($p = 1$)	Membranes ($p \geq 0$)
(1) Particles have a one-dimensional *worldline* $\gamma : [s_0, s_1] \to M$.	(1) Membranes have a p-dimensional *worldvolume* $\gamma : \Sigma \to M$. (Called a *world-sheet* for $p = 2$, or a particle that exists only for an instant if $p = 0$... These latter (for $p = 0$) are called *instantons*.)
	
(2) The extended phase space is T^*M (which keeps track of momentum too). To get this, look at the tangent to our curve γ, and take the dual space, so we get $(TM)^*$.	(2) What is the extended phase space here? Here, the tangent "sheet" consists of p-wedges (wedges for orientation reasons) of tangent vectors, and dualizing gives $(\bigwedge^p TM)^*$.
(3) We start studying $\widetilde{\gamma} : [s_0, s_1] \to T^*M$, because T^*M has a canonical 1-form $\widetilde{\alpha}$ on it.	(3) We start studying $\widetilde{\gamma} : \Sigma \to \bigwedge^p(T^*M)$, because it has a canonical p-form on it. (We'll check this later.)

8.1. **More derivations.** Recall that our worldline has a tangent vector $\gamma'(s) \in T_{\gamma(s)}M$, called its velocity. (Also, we need this velocity - or even the analogue for general p - to vary continuously/smoothly, like a section.) To generalise this to general p, one option is that $\gamma : \Sigma \to M$ gives $d\gamma : T\Sigma \to TM$ (or $d\gamma(x) : T_x\Sigma \to T_{\gamma(x)}M$).



$\in \; T_{\gamma(s)}M$

FIGURE 10. tangent vector

But over in this situation, we actually want our answer to lie in $\bigwedge^p TM$. Thus, let us (also) call the analogue $\gamma'(x) \in \bigwedge^p TM$, called its *multi-velocity* or *p-velocity*, which says how fast, *and in which direction* (of coordinates on

Σ), γ moves. For example, for $p = 2$, the picture looks like the following "parallelogram" or area therein, inside $\bigwedge^2 T_{\gamma(x)}M$:

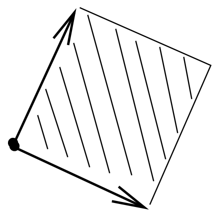


FIGURE 11. shaded parallelogram = determinant

Here's one possible definition of the multi-velocity for general p: let's define γ' using some choice of local coordinates $s_1, \ldots, s_p$ on Σ:

$$\gamma'(x) = \frac{\partial\gamma(x)}{\partial s_1} \wedge \cdots \wedge \frac{\partial\gamma(x)}{\partial s_p} \in \bigwedge{}^p T_{\gamma(x)}M$$

Remark 8.1.

(1) Another choice of local coordinates s_i should give a "rescaled" section.
(2) This definition *is* a section, as we will eventually see. For now, we look at it only at a point $\gamma(x) \in M$.
(3) This vanishes if the $\frac{\partial\gamma(x)}{\partial s_i}$'s are linearly dependent (i.e. γ is not an immersion). For instance, if a cylinder is shrunk to a line (or in general, $\dim M < p$ etc.).
(4) The challenge here, is to invent a coordinate-free definition of γ'.

8.2. **Generalizing the Lagrangian.** Earlier, we had $L : TX \to \mathbb{R}$. This is not good, since we want eventually to forget about X, and only use $M = X \times \mathbb{R}$. Thus, we generalize this notion, and consider general $L : TM = T(X \times \mathbb{R}) \to \mathbb{R}$. (The earlier notion of Lagrangian is, now, merely independent of time.)

This is a good thing to do because now we have $(\gamma(s), \gamma'(s)) \in TM$, i.e. velocity is really in TM.

To generalize to p-membranes, we now want to consider the Lagrangian to be a function $L : \bigwedge^p TM \to \mathbb{R}$, since now, $(\gamma(x), \gamma'(x)) \in \bigwedge^p TM$.

Question. Why is space-time still $X \times \mathbb{R} = M$ here, instead of $X \times \Sigma$
One possible reason: Σ is not "all-time-coordinates", so to speak; it's just "one-time-and-$(p-1)$-space" coordinates!

9. Nov 28, 2006: More on particles "vs." membranes

Recall from last time, that we were trying to generalize our framework from $p = 1$ to general p. This resulted in our changing the base parameter space $[s_0, s_1]$ to a p-dimensional manifold Σ, possibly with boundary (and corners).

9.1. (Functorial) construction of the multivelocity alternating tensor.

Here's how to define the *multivelocity* γ' of a map $\gamma : \Sigma \to M$, i.e. of a p-dimensional membrane in the space-time M:

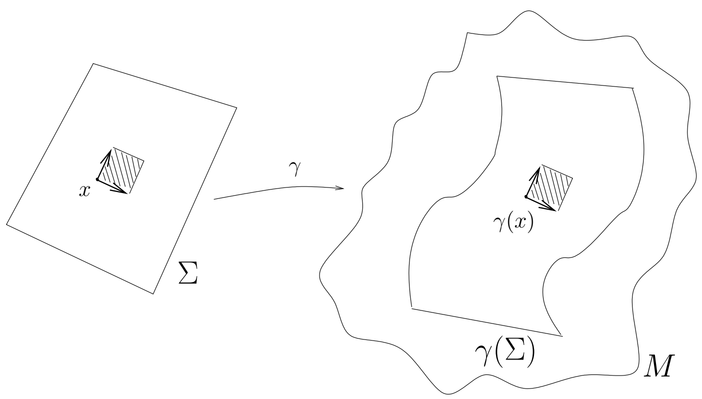


Figure 12. $[s_0, s_1]$ maps to a tangent at a point on a path γ; Σ maps to a shaded parallelogram at a point on a surface M

To define γ' in a coordinate-free way, take $\gamma : \Sigma \to M$ and apply various functors to get γ' (if we want to be really precise!).

(1) *The tangent functor:* $\gamma : \Sigma \to M$ goes to its differential $d\gamma : T\Sigma \to TM$, or to be very precise, the tangent functor $T\gamma : T\Sigma \to TM$. Here, T is the *tangent functor* that takes manifolds to vector bundles over them (more precisely, $T : M \mapsto TM$), and morphisms between manifolds, to their differentials, which are vector bundle maps between the corresponding tangent bundles.

(2) *The (top-)wedge functor:* We now apply the functor $\bigwedge^p$ that takes finite-rank vector bundles on M to finite-rank vector bundles on M. Thus, if $(E \to M)$ has fiber E_x of dimension m over $x \in M$, then it is mapped to $\bigwedge^p(E \to M) = (\bigwedge^p E) \to M$, with fiber $(\bigwedge^p E)_x = \bigwedge^p E_x$ of dimension $\binom{m}{p}$ for all x. In particular, if $m = p$, then we get a line bundle on M, also called the *determinant line bundle.*

How about morphisms? Given a vector bundle map $f : E \to E'$ (both E, E' are over M), we get $\bigwedge^p f : \bigwedge^p E \to \bigwedge^p E'$, given by

$$(\bigwedge{}^p f)_x(e_1 \wedge \cdots \wedge e_p) := f(e_1) \wedge \dots f(e_p), \ \forall e_i \in E_x$$

9.2. **Volume forms.** Since we are working over Σ, hence we thus get, given $\gamma : \Sigma \to M$, the map

$$\bigwedge{}^p d\gamma : \bigwedge{}^p T_x\Sigma \to \bigwedge{}^p T_{\gamma(x)}M$$

But the multivelocity was just *one* vector inside it, not a function! Actually, the domain above is just one-dimensional, being the top exterior power, but this means that there still is the choice of a scalar involved. Moreover, this scalar must be nonzero for any x (so that our domain vector is nonzero), which means that we are looking for a nonvanishing section of the top exterior power.

Definition 9.1. A *volume form* is a nonvanishing section of the top exterior power.

For example, $\int(-)dx \wedge dy \wedge dz$ takes in functions and spits out numbers.

So, *assume* for now, that Σ comes equipped with a volume form vol $\in \Omega^p(\Sigma)$, so that $0 \neq \mathrm{vol}_x \in \bigwedge^p T_x^*\Sigma$ for all $x \in \Sigma$. Then the top (pth) exterior powers of the tangent and cotangent bundles on Σ are both of rank one, hence the evaluation map on their fibers gives an isomorphism between them. More precisely, if $V_x = \bigwedge^p T_x\Sigma$, then $\dim V_x = \dim V_x^* = 1$, where $V_x^* = \bigwedge^p T_x^*\Sigma$. One then defines $\varphi_x : V_x^* \xrightarrow{\sim} V_x$, by $\varphi_x(0) = 0$, and

$$\langle \varphi_x(\omega), \omega \rangle = 1 \ \forall 0 \neq \omega \in V_x^*$$

Warning. Note that the map φ_x is not linear. It is "inverse-linear".

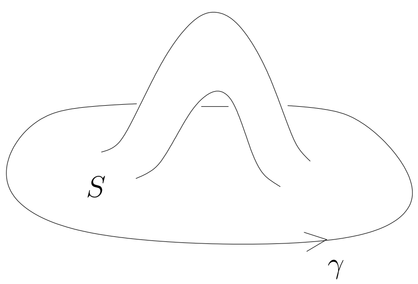
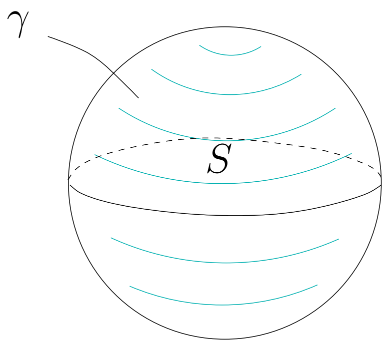
We can now define the multi-velocity.

Definition 9.2. Given a volume form vol $\in \Omega^p(\Sigma)$ and $\gamma : \Sigma \to M$, define the *multi-velocity* γ' at $\gamma(x) \in M$, to be

$$\gamma'(x) = (\bigwedge{}^p d\gamma)(\varphi^{-1}(\mathrm{vol}_x)) \in \bigwedge{}^p T_{\gamma(x)}M$$

Remark 9.3. The crucial fact is that this definition is *not* independent of vol! However, what we will see later, is that the notion of the *volume* (that generalizes the length of a string and area of a membrane) is now independent of choice of vol (i.e. choice of normalization).

We once again develop the two pictures side-by-side, for particles and for general membranes.

Particles ($p = 1$)	Membranes ($p \geq 0$)
(1) A particle comes equipped with a *worldline* $\gamma : [s_0, s_1] \to M =$ space-time.	(1) A membrane has a *world-volume* $\gamma : \Sigma \to M$.
(2) Then comes a *Lagrangian* and an *action*. For instance, in general relativity, M is a Lorentzian manifold, and the action for a particle with mass m and charge e, is $S(\gamma) = m\ell(\gamma) + e\int_\gamma A$ where $A \in \Omega^1(M)$ is the "electromagnetic vector potential", and ℓ is the length, obtained using the metric on M: $\ell(\gamma) = \int_{s_0}^{s_1} \|\gamma'(s)\| \, ds$	(2) The usual *Nambu-Goto* action for a membrane is $S(\gamma) = m \,\mathrm{Vol}(\gamma) + e\int_\gamma A$ where e is the charge, m is the *membrane tension* now, and A is the *p-form potential*. Finally, Vol stands for the volume of γ, also obtained using the metric on M: $\mathrm{Vol}(\gamma) = \int_\Sigma \|\gamma'(x)\| \,\mathrm{vol}_x$
(3) (Marginal cases.) If $A = 0$, a particle extremizing the action traces out a *geodesic*.	(3) (Marginal cases.) If $A = 0$, a membrane extremizing the action traces out a *minimal surface*.
(4) On the other hand, if m or the metric vanishes, i.e. we consider only the electromagnetic part of the action, namely $e\int_\gamma A$, then this gives a change of phase $\exp(ie\int_\gamma A) \in U(1) = S^1$, when we move a *quantum particle* along a path γ. In fact, if γ is a loop *homologous* to the trivial loop, and it "bounds a surface" S, then by Stokes' Theorem, $\int_\gamma A = \int_S F$, where $\partial S = \gamma$, and $dA = F$ is the *electromagnetic field*.	(4) If we do the same thing here, i.e. consider only the p-form potential, then we get a change of phase $\exp(ie\int_\gamma A) \in U(1)$, when we move a *quantum membrane* along the surface γ. In fact, if γ is a surface homologous to the trivial one, and it bounds a higher-dimensional manifold S, then by Stokes' Theorem, $\int_\gamma A = \int_S F$, where $\partial S = \gamma$ and $F = dA$.
	

Remark 9.4.

(1) In the Nambu-Goto action above, A is the p-form generalization in Maxwell's equations, hence is also called *p-form electromagnetism.*
(2) Also, $|\gamma'(x)|$ uses the metric on $\bigwedge^p TM$, which in turn is obtained from the inner product/metric on TM. Moreover, here we use our chosen volume form vol on Σ.
(3) Then the formula for the volume is independent of choice of vol! This is because any scaling of $\gamma'(x)$ involves a scaling inside φ (by the definition above), which was "inverse-linear", and the corresponding choice of vol_x is rescaled by exactly the reciprocal of the earlier scaling. This makes the entire expression independent of choice of normalization.
(4) Also note that the integral had better converge. Thus, we need to have some restriction on Σ, or else at least on γ. For example, the image of γ may need to be compact in M.
(5) In either of these two cases, $e \int_\gamma A$ is independent of parametrization as well. Therefore the entire expression in the formula above, is also independent of parametrization, as we wanted.
(6) Finally, we get a map $C_\gamma : \Omega^p(M) \to U(1) = S^1$ for each such γ, namely: $C_\gamma(A) = \exp(i \int_\gamma A)$. (Here, we're arranging the units so that $\hbar = 1$.) This brings us closer to cohomology, which we will in fact encounter next time.

9.3. **The canonical p-form.** We now do the computations similar to the $p = 1$ case above. Let us denote the canonical p-form on $\bigwedge^p T^*M$ by α_p. Since we will use (q^i, p_i) for coordinates on the cotangent bundle, let us compute instead, the *canonical n-form* α_n on $\bigwedge^n T^*M$.

Using coordinates: An element of $\bigwedge^n T^*M$, at a point $q \in M$, has a "Plücker" basis p_I, where $I = \{i_1 < i_2 < \cdots < i_n\} \subset \{1, 2, \dots, \dim M\}$. Define corresponding differentials $dq^I := dq^{i_1} \wedge \cdots \wedge dq^{i_n}$, and the n-form by

$$\alpha_n := p_I dq^I$$

where we still use Einstein summation for index sets - or for n-tuples of indices.

Coordinate-free definition: As earlier, keep in mind that α_n likes to eat n-wedges of tangent vectors to T^*M, i.e.

$$v_1 \wedge \cdots \wedge v_n, \ v_i \in T_{(q,p)}(T^*_q M)$$

and spit out numbers. But we have the projection π_1 from T^*M, taking (q, p) to q. Thus, $d\pi_1(v_i) : T_{(q,p)}(T^*M) \to TM$. Moreover, $\pi'_2(v_i) \in T^*M$ (as earlier), so we now define

$$\alpha_n(\wedge_i v_i) = (\wedge_i \pi'_2(v_i))(\wedge_i d\pi_1(v_i))$$

As above, we have

Theorem 9.5. *If q^i are any local coordinates on M, and (q^i, p_i) the corresponding local coordinates on T^*M, then $\alpha_n = p_I dq^I$.*

Proof. (This is a (somewhat tortuous!) exercise in keeping track of multi-indices in wedging, so feel free to skip it!) Choose any $v_i \in T_{(q,p)}T^*_q M$ for $1 \le i \le n$. Then, as earlier,

$$\wedge_i v_i = \sum_{K,J} a^K_J \wedge_{k\in K} \frac{\partial}{\partial q^k} \bigwedge \wedge_{j\in J} \frac{\partial}{\partial p_j}$$

for some choice of scalars a^K_J, and where K, J vary over disjoint subsets of $\{1, 2, \ldots, \dim M\}$, so that $|K| + |J| = n$. In fact, if $v_i = \sum_l a^{l,i} \frac{\partial}{\partial q^l} + a_{l,i} \frac{\partial}{\partial p_l}$, then

$$a^K_J = \sum_{i_k, i'_j} \pm \prod_{k\in K} a^{k,i_k} \cdot \prod_{j\in J} a_{j,i'_j}$$

where we sum over all tuples i_k and all i'_j, so that they are all distinct, and the $\pm$ is the product of the signs of the two permutations, that rearrange the sets $\{i_k : k \in K\}$ and $\{i'_j : j \in J\}$ in increasing order.

As above, we also condense our notation by writing

$$\frac{\partial}{\partial q^K} := \wedge_{k\in K} \frac{\partial}{\partial q^k}, \qquad \frac{\partial}{\partial p_J} := \wedge_{j\in J} \frac{\partial}{\partial p_j}$$

where all wedges are taken with indices in increasing order (just as in dq^I, or even in p_I). Now by our "random facts" on wedges, we know that

$$dq^I\left(\frac{\partial}{\partial q^K} \wedge \frac{\partial}{\partial p_J}\right) = \delta^I_K \delta^J_\emptyset$$

since $|I| = n = |K| + |J|$. In other words, it vanishes if J is nonempty, or if $J = \emptyset$ (so $|I| = |K|$) but $I \neq K$.

Method 1: We therefore start our computations. Firstly, evaluating $\wedge_i v_i$ at $p_I dq^I$, we have

$$p_I dq^I(\wedge_i v_i) = \sum_{J,K} p_I a^K_J \delta^I_K \delta^J_\emptyset = p_I a^I_\emptyset$$

where most terms in the sum vanish because of the Kronecker deltas.

Method 2: We now want to reconcile this to the coordinate-free definition in these coordinates. In this case, we note that if

$$v = \left(q, p, \ \sum_i a^i \frac{\partial}{\partial q^i} + a_i \frac{\partial}{\partial p_i}\right)$$

then $d\pi_1 : T(T^*X) \to TX$ sends it to $(q, \sum_i a^i \frac{\partial}{\partial q^i})$, where q^i's on the right-hand side are local coordinates in a different space (TX) than the q^i's on the left-hand side. In other words, $d\pi_1(v_i)$ removes the vertical components $\frac{\partial}{\partial p_j}$'s from each v_i, and takes the $\frac{\partial}{\partial q^i}$'s to themselves in the tangent space to the vector space $T^*_q X$.

Therefore, $\wedge_i d\pi_1(v_i)$ precisely equals $a_\emptyset^K \frac{\partial}{\partial q^K}$, since there are no vertical components remaining!

Moreover, as in the earlier proof (for $p = 1$), we have that $\pi_2(v) = p_i dq^i$ in the cotangent space at q. Wedging this over all v_i's gives us all possible terms of type $p_I dq^I$. Thus, we can now compute α_n using the coordinate-free definition, to get

$$\alpha_n(\wedge^i v_i) = p_I a_\emptyset^K dq^I(\frac{\partial}{\partial q^K}) = \delta_K^I p_I a_\emptyset^K = p_I a_\emptyset^I$$

and since the two definitions agree at all $\wedge_i v_i$, hence the forms themselves are equal, as desired. □

10. Dec 05, 2006: Phases and connections on bundles

Today we shall return to our extended phase space, so to speak. From now on, we forget about X, and only talk about our *phase space* M. Thus, the paths and 1-forms are henceforth denoted by γ, α etc. (and not $\widetilde{\gamma}, \widetilde{\alpha}$ etc.).

A glimpse of what is to come: Having reviewed classical particle mechanics, and its stringy and brany generalizations, next we will quantize these. We'll mainly talk about geometric quantization.

10.1. When connections come in.

In particle mechanics, a phase space is often a symplectic manifold (M, ω) (but we'll see an example today, that is not of this form). The role of the 2-form ω is to (help) compute the action of a path $\gamma \in M$: $S(\gamma) = \int_\gamma \alpha$, where $d\alpha = \omega$ (upto a sign, which we shall henceforth ignore, since we do not care about reconciling this to physicists' notation anymore!).

What about ω itself? If $\gamma = \partial\Sigma$ (i.e. γ is homologous to the boundary of a surface Σ - always assumed to be *compact* here, in order to integrate on it), then $S(\gamma) = \int_\Sigma \omega$ by Stokes' Theorem.

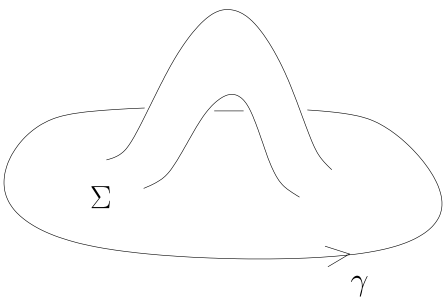


Figure 13. γ bounds Σ that might have handles on it

This raises two questions:

(1) What if γ is a path not cohomologous to zero, i.e. $\gamma \neq \partial\Sigma$?
(2) What if $\omega \neq \partial\alpha$ for any α? (For cotangent bundles, this problem does not arise, but in general symplectic manifolds, ω is merely closed, not exact.)

(One obvious solution is: if γ is cohomologous to zero but ω is not exact, we use $S(\gamma) = \int_\Sigma \omega$ - and if γ is not cohomologous to zero but ω is exact, then $\omega = d\alpha$, and we use $S(\gamma) = \int_\gamma \alpha$. Thus, we only get in trouble if both these problems arise simultaneously.)

But we look here, at a generalization of a 1-form that works, called a *connection on a $U(1)$-bundle.*

10.2. **Phases and relative phases.** In quantum mechanics, what matters is not the action $S(\gamma) \in \mathbb{R}$, but the *phase* $e^{iS(\gamma)} \in U(1)$. Note that this already loses information, being periodic:

$$0 \to 2\pi\mathbb{Z} \to \mathbb{R} \xrightarrow{s \mapsto e^{is}} U(1) \to 0$$

Actually, if one has (in quantum mechanics) a bunch of paths between points m_0, m_1 in M, i.e.

$$\wp \subset P_{m_0 \to m_1} M := \{\gamma : [s_0, s_1] \to M : \gamma(s_i) = m_i\}$$

(depending on what we need, γ might need to be differentiable, smooth, etc.), then the *amplitude* to get from m_0 to m_1 (via $\wp$) is

$$A_\wp := \int_{\gamma \in \wp} e^{iS(\gamma)} \; D\gamma \in \mathbb{C}$$

(modulo convergence issues, of course). Such an expression is called a *path integral* (it is one of the two major types of Feynman integrals).

The amplitude now gives a probability $|A_\wp|^2 \in [0, 1]$, upto a suitable renormalization.

So, for physics, changing $e^{iS(\gamma)} \mapsto ce^{iS(\gamma)}$ for some $c \in U(1)$, does not change any probabilities.

However, a connection on a $U(1)$-bundle lets us calculate a definite phase (i.e. an actual, concrete value) for any *loop* γ, and thus a ratio of phases for any pair of paths γ_1, γ_2 from m_0 to m_1, as follows:

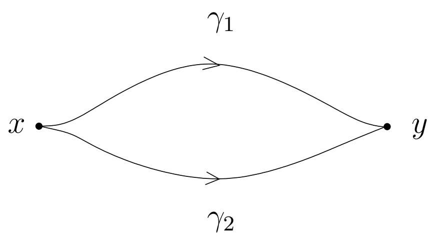


FIGURE 14. two paths $\gamma_1, \gamma_2 : m_0 \to m_1$; maps to the loop $\gamma_1 \circ \gamma_2^{-1}$

We define $\dfrac{\text{Phase}(\gamma_1)}{\text{Phase}(\gamma_2)} = \text{Phase}(\gamma)$, where γ is the loop $\gamma_1 \circ \gamma_2^{-1}$ (based at m_0, so note the order of composing). These *relative phases* are what we can actually measure: they are unchanged by multiplying $\text{Phase}(\gamma_1), \text{Phase}(\gamma_2)$ by any $c \in U(1)$.

Next, any $U(1)$-connection has a *curvature 2-form* $\omega \in \Omega^2(M)$, which should be our symplectic structure in applications to geoemtric quantization. If γ is a loop with $\partial\Sigma = \gamma$ for some surface Σ, then

$$\text{Phase}(\gamma) = e^{i \int_\Sigma \omega}$$

This is just like $e^{i\int_\gamma A} = e^{i\int_\Sigma \omega}$, but now Phase($\gamma$) is not coming from a 1-form A anymore.

Warning. Before we move on to a concrete example, let us remark that not every loop is the boundary of a (compact) surface Σ. For example, consider the unit circle in the plane punctured at the center of the circle. Even as topological spaces, the boundary of the punctured disk is the union of a circle and a point, not just the circle. And as manifolds with boundaries, the point is not even of codimension one.

10.3. **Example: Rigid rotor.** Let $M = S^2 \subset \mathbb{R}^3$ be the unit sphere, with the 2-form $\omega = \sin^2\phi \; d\theta \; d\phi$, where θ, ϕ denote the longitude, and the latitude from the north pole (not equator) respectively.

[figure: sphere with equatorial plane and angle θ; prime meridian and angle ϕ with north pole]

Then $\int_\Sigma \omega = \text{area}(\Sigma)$.

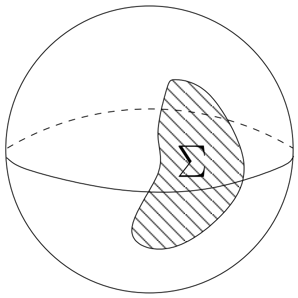


FIGURE 15.

What's our $U(1)$ connection here? Given a loop γ, pick any unit tangent vector v at a point x, and parallel transport it around γ, to get a new tangent vector v'.

Thus, v' is obtained by rotating v in the tangent plane at x, and this gives Phase(γ) $\in U(1)$. Moreover, in this case we really do have

$$\text{Phase}(\gamma) = e^{i\int_\Sigma \omega} = \text{area}(\omega)$$

when $\partial\Sigma = \gamma$. We say that ω is the curvature of our $U(1)$-connection.

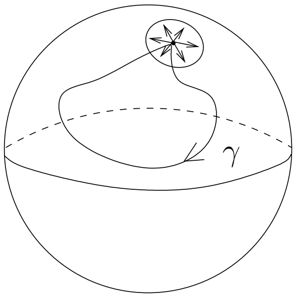


FIGURE 16. shows the tangent plane at a point on a sphere, the loop γ, and the vectors v, v'

M is then the phase space (not configuration space) of a *rigid rotor* i.e. a rigid spinning ball with fixed rate of rotation (angular speed), but unknown axis and "direction of spinning".

Remark 10.1.

(1) Note that the ambiguity of antipodal points giving the same axis of rotation is removed, if we decree that all rotations are specified by the north pole of the axis, and the right-hand-rule to determine the direction of rotation.
(2) M is also the Riemann sphere, and the phase space of a cubit. The quantization of this is the phase space of an electron.
(3) Also note that M is the phase space of a rigid rotor, because both position and velocity are mentioned here.

Note that M is 2-dimensional, so the 2-form ω is obviously closed. However, we claim that M is not the cotangent bundle of any manifold. In fact,

Proposition 10.2. *ω is not exact.*

Proof. Suppose not, i.e. say $\omega = d\alpha$ for some 1-form α. Then if $\partial\Sigma = \gamma$ for some (connected) compact region Σ on S^2, denote Σ' to be the closure of the complement $S^2 \setminus \Sigma$. This too has boundary γ, but perhaps with the opposite orientation. In either case, we now use Stokes' Theorem to get

$$\text{area}(\Sigma) = \int_\Sigma \omega = \int_\gamma \alpha = \pm \int_{\Sigma'} \omega = \pm(\text{area}(S^2) - \text{area}(\Sigma))$$

for all Σ as above. This is a contradiction (e.g. take $\Sigma = S^2,\ \Sigma' = \emptyset$). $\square$

10.4. **Integral cohomology and Max Planck.** More generally, say (M, ω) is any symplectic manifold equipped with a $U(1)$-connection whose curvature is ω, and say Σ is a surface inside M. Then $\text{Phase}(\gamma) = e^{i\int_\Sigma \omega}$ when $\partial\Sigma = \gamma$.

Now consider two surfaces Σ_1, Σ_2 with common boundary γ - or more precisely, $\partial\Sigma_1 = \gamma,\ \partial\Sigma_2 = \gamma^{-1}$. Let $\Sigma = \Sigma_1 \cup \Sigma_2$.

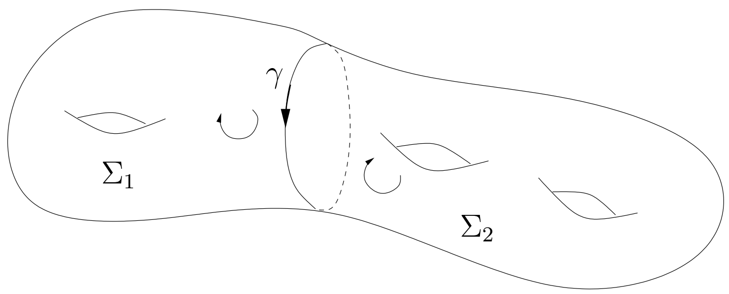


FIGURE 17. a surface (say with genus) obtained by joining two half-surfaces

Then we have

$$\mathrm{Phase}(\gamma) = e^{i\int_{\Sigma_1}\omega} = e^{i-\int_{\Sigma_2}\omega}$$

so that if Σ is as above, then

$$e^{i\int_{\Sigma}\omega} = e^{i\int_{\Sigma_1}\omega + \int_{\Sigma_2}\omega} = 1$$

Therefore $\int_{\Sigma}\omega \in 2\pi\mathbb{Z}$!

This argument works for any $\Sigma \subset M$, with $\partial\Sigma = \emptyset$, since we can always write $\Sigma = \Sigma_1 \cup \Sigma_2$ as in the picture above. So, ω describes an *integral* second cohomology class.

Remark 10.3.

(1) This is one origin of the word "quantization", as Max Planck and Bohr-Sommerfeld essentially realised. The energy states have to occur with some integer-conditions...

(2) We'll see later (next quarter!) that this generalizes to $(p+1)$-forms for all p!