A Spring in Imaginary Time

Jeff Morton

1. If we have a spring with fixed ends tracing a curve q in $\mathbb{R}^n$ whose energy is E as given, we find that taking the variation of E gives:

$$\begin{aligned}
\delta E &= \delta \int_{s_0}^{s_1} \left(\frac{k}{2}\dot{q}(s)\cdot\dot{q}(s) + V(q(s))\right) ds \\
&= \int_{s_0}^{s_1} \left(\frac{k}{2}\delta(\dot{q}(s)\cdot\dot{q}(s)) + \delta V(q(s))\right) ds \\
&= \int_{s_0}^{s_1} \left(k\dot{q}(s)\cdot\delta\dot{q}(s) + \nabla V(q(s))\cdot\delta q(s)\right) ds \\
&= \int_{s_0}^{s_1} \left(-k\ddot{q}(s) + \nabla V(q(s))\right)\cdot\delta q(s) ds \qquad \text{(by parts)}
\end{aligned}$$

The boundary terms from the integration by parts disappear since we only consider variations which fix the endpoints of the curve traced by the spring (i.e. $\delta q = 0$ at s_0 and s_1). Then if we have that $\delta E = 0$ for all variations δq, then the equation q must satisfy is

$$-k\ddot{q}(s) + \nabla V(q(s)) = 0$$

or

$$k\ddot{q}(s) = \nabla V(q(s))$$

2. If $V = mgz$ in $\mathbb{R}^3$, we have that $\nabla V = (0, 0, mg)$, so that $\ddot{q}(s) = (0, 0, \frac{mg}{k})$. That is, the curve has a constant positive acceleration $\frac{mg}{k}$ in the z direction with respect to the parameter s, and constant velocity in the x and y directions with respect to s. So we can also think of the curve as having constant acceleration in the z direction with respect to distance in the (x, y) direction of the particle's horizontal velocity. So the curve is a parabola with local maxima at the endpoints.

3. Replacing s by it, we get

$$\begin{aligned}
E &= \int_{it_0}^{it_1} \left(\frac{k}{2}\dot{q}(it)\cdot\dot{q}(it) + V(q(it))\right) d(it) \\
&= i\int_{it_0}^{it_1} \left(\frac{k}{2}q(it)\cdot\frac{d}{d(it)}q(it) + V(q(it))\right) dt \\
&= i\int_{it_0}^{it_1} \left(\frac{i^2k}{2}\dot{q}(it)\cdot\dot{q}(it) + V(q(it))\right) dt \\
&= -i\int_{t_0}^{t_1} \left(\frac{k}{2}\|\dot{q}(-t)\|^2 - V(q(-t))\right) dt \\
&= -i\int_{-t_0}^{-t_1} \left(\frac{k}{2}\|\dot{q}(t)\|^2 - V(q(t))\right) dt
\end{aligned}$$

Indeed, this is just $-i$ multiplied by the action along a path of a particle moving in a potential ($K - V$, where $K = k/2\|\dot{q}\|^2$ with k playing the role of the mass m).

4. We have the analogy:

STATICS	DYNAMICS
Principle of Least Energy	Principle of Least Action
spring	particle
energy	action
stretching energy	kinetic energy
potential energy	potential energy
spring constant k	mass m

5. The statics problem in (2) corresponds to the dynamics problem of a particle moving in a potential with constant gradient. The solution to that problem has the particle moving in a parabola with local *minima* at the endpoints - the acceleration is in the direction opposite to that observed in the statics problem of the spring.

6. Formally replacing t by $\boldsymbol{t}$ in Newton's equation $F = ma$, where $a(t)$ is $\ddot{x}(t)$, the second derivative of position with respect to t:

$$\begin{aligned} F &= m\frac{d^2}{d\boldsymbol{t}^2}x(\boldsymbol{t}) \\ &= mi^2\ddot{x}(\boldsymbol{t}) \\ &= -m\ddot{x}(\boldsymbol{t}) \end{aligned}$$

(This equation $F = -m\ddot{x}$ is reminiscent of Hooke's law for springs, except that "acceleration" $\ddot{x}$ plays the role of displacement.)