Quantum Gravity Seminar
Homework 5

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Show that if α is the canonical 1-form on T^*X and $\omega = -d\alpha$, then ω is nondegenerate.

We call ω nondegenerate if $\forall v \neq 0$, $\exists u$ such that $\omega(v,u) \neq 0$.
Let $v = a^i \frac{\partial}{\partial q^i} + b_i \frac{\partial}{\partial p_i} \neq 0$.

$$\begin{aligned}
\omega(v,-) &= dq^i\left(a^i\frac{\partial}{\partial q^i} + b_i\frac{\partial}{\partial p_i}\right)dp_i - dp_i\left(a^i\frac{\partial}{\partial q^i} + b_i\frac{\partial}{\partial p_i}\right)dq^i \\
&= \left(a^i dq^i\frac{\partial}{\partial q^i} + b_i dq^i\frac{\partial}{\partial p_i}\right)dp_i - \left(a^i dp_i\frac{\partial}{\partial q^i} + b_i dp_i\frac{\partial}{\partial p_i}\right)dq^i \\
&= a^i dp_i - b_i dq^i
\end{aligned}$$

Since $v \neq 0$, there exists an index k such that $a^k \neq 0$ or $b_k \neq 0$. Then let $u = c^i \frac{\partial}{\partial q^i} + d_i \frac{\partial}{\partial p_i}$

$$\begin{aligned}
\omega(v,u) &= a^i dp_i\left(c^i\frac{\partial}{\partial q^i} + d_i\frac{\partial}{\partial p_i}\right) - b_i dq^i\left(c^i\frac{\partial}{\partial q^i} + d_i\frac{\partial}{\partial p_i}\right) \\
&= a^i d_i - b_i c^i
\end{aligned}$$

So if $a^k \neq 0$, let $c^i = 0$ for every i and $d_i = \delta^i_k$. And a similar situation when $a^k = 0$.
Thus $\omega(v,u) \neq 0$.