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Quantum Gravity Seminar
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2007-08 Module on Geometric Representation Theory

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Introduction, Overview, Preview

(1) (a) In the "classical" world, we study actions of, say, groups, on, say, sets. Formally, we say

G = category with one object $\cdot$, and all morphisms invertible.

Set = category of sets, morphisms = functions.

and then a G-set X is merely a functor

$$R_X : G \to \mathrm{Set}$$
$$\cdot \mapsto X$$

(b) In the "quantum" world, we have interesting things that come up, e.g. we have superposition / or adding of vectors.

So, here we replace Set by $\mathbb{C}$-Vect or $\mathrm{Vect}_{\mathbb{C}}$ and then morphisms are just linear operators.

Then a G-action is ~~[illegible]~~ but a G-representation.

(c) Well, geometric representation theory aims to understand (b) via (a). One way to continue it is to search for a completion to the picture

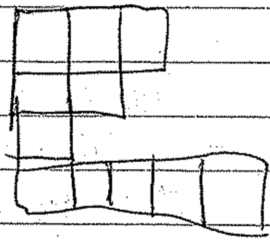
$$\begin{tikzcd} G \arrow[r] \arrow[dr] & \mathrm{Set} \\ & \mathrm{Vect}_{\mathbb{C}} \end{tikzcd}$$

i.e. a functor : $\mathrm{Set} \to \mathrm{Vect}_{\mathbb{C}}$ with —

One option is the Free Construction Functor:

$X \mapsto \mathbb{C}^X$ = vect-space with basis X.

eg:



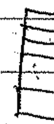
$n_1 = 3$
$n_2 = 2$ $\quad n = 11$
$n_3 = 1$
$n_4 = 5$

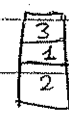
(2) A <u>(combed) (n-box) Young diagram</u> is one where $n_1 \geq n_2 \geq \cdots$

(3) Given an uncombed n-YD (we'll henceforth call it $UYD(n)$)
$(n_1, \ldots, n_k) = D$, a <u>D-flag</u> on an n-element set X is

$$\emptyset = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_k = X, \text{ with } |X_i \setminus X_{i-1}| = n_i \ \forall i$$

(4) Let us denote the set of ~~n-fl~~ D-flags on $[n] = n$, by $D(n)$

eg: if $D =$  $\} n$. $\Rightarrow$ There are $n!$ D-flags on n, eg



etc.

The main result here is:

Theorem: Every irrep of $n!$ is a sub-representation of the rep. of $n!$ on $\mathbb{C}^{D(n)}$ for some uncombed n-box YD D.

In fact, it's enough to use only combed ones.
<u>AND</u>: each such D has its "special" irrep inside it!
So, these cover <u>all</u> irreps of $n!$

(It's called "n-factorial" or "n-bang")

n!, so, we'll make sense set-theoretically / categorically of things like $\binom{n}{k}$ etc!

So WARNING: $\mathbb{C}^X \neq \mathrm{Fun}(X, \mathbb{C})$ for infinite X

(almost all entries are 0)

② Examples ($n = [n]$ is a fixed n-element set)

① (i) The symmetric group $S_n =: (n!)$ acts on n, hence on $\mathbb{C}^n$

$3!$ acts on $\mathbb{C}^3$, and note: $\mathbb{C}^n$ is not a $n!$-irrep unless $n=1$

($\mathbb{C}^3 \cong \mathbb{C}^1 \oplus \mathbb{C}^2$ as $3!$-reps)

So, in general, we can't get irreducible modules.

(ii) (Note: For G finite, irreducible = indecomposable in $\mathbb{C}G$-mod: Maschke's Theorem)

However, one theme of this course is Hecke operators which will allow us to get irreps.

(iii) If G acts on X and $S(X)$ is some set of structures that we can put on X, then G acts on $S(X)$.

Eg. $S(X)$ = set of 2-colourings on X
ie every $x \in X$ is either red or blue

We'll get lots of actions of $n!$ from structures on n-element sets.

Here's an example:

Defn ① An uncombed n-box Young diagram is a list $n_1, \dots, n_k$ of positive integers, that add up to n.

Examples (2):

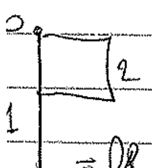
For any field F, GL(n, F) has a (natural?) representation on F^n. There's a unique (upto isom.) with q elements, denoted by $\mathbb{F}_q$, iff $q = p^n$ for some prime p, and $n \geq 1$.

Let D be an uncombed n-box. A D-flag on F^n is

$$\{0\} = X_0 \subseteq X_1 \subseteq \dots \subseteq X_k = F^n, \text{ such that}$$
$$\dim(X_i) - \dim(X_{i-1}) = n_i$$

Example: $D = \square \}n$ | How many D-flags are ~~there~~ on $\mathbb{F}_q^n$?

for n = 3,



= flag!

ANS: First, $X_0 = \{0\}$ ✓

$$\#(X_1) = \# \text{ lines in } \mathbb{F}_q^n$$
$$= \frac{q^n - 1}{q-1} \begin{array}{l} \to = \text{other pt on line} \\ \to = \text{rescalings} \end{array}$$

$\# X_2$ given such an X_1

$$= \# \text{ lines in } \left(X_k = \mathbb{F}_q^n\right) / X_1 \, !$$

$$\text{So, } = \frac{q^{n-1} - 1}{q-1}$$

⋮

$$\text{So, } \# D\text{-flags} = \frac{q^n - 1}{q-1} \cdot \frac{q^{n-1} - 1}{q-1} \cdots \frac{q - 1}{q-1}$$

(b) <u>Now note</u>: **Def'n** The <u>quantum integers</u> are $[n]_q = \frac{q^n - 1}{q-1}$

and ~~the~~ the <u>quantum factorials</u> are $[n]_q! = \prod_{i=1}^{n} [i]_q$.

So, the answer here is $[n]_q!$

Thus, ans. in Example (1) $= q \to 1$ limit of ans. in Example (2)!

This gives us an idea of <u>a field with one element</u>:

$$\boxed{\mathbb{F}_1^n = \text{finite sets} \;!}$$

— X —

(c) <u>Homework</u>: ~~The~~ The above result holds for <u>any</u> D: the # D-flags on $[n]$

$= q \to 1$ limit of # D-flags on $\mathbb{F}_q^n$.

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(3) <u>Projective geometry</u>

e.g. $\mathbb{R}P^2 = \{1\text{dim'nal subspaces of } \mathbb{R}^3\}$

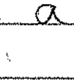
(a) There is a nice duality between points and lines in projective geometry:

Any two lines ~~dot~~ in projective space determine a unique point, and vice versa.

(b) Thus, $\mathbb{F}P^{n-1} = \{1\text{-dim subspaces of } \mathbb{F}^n\}$

Also, a line in $\mathbb{F}P^{n-1}$ = "plane" in $\mathbb{F}^n$, etc.

A D-flag in $\mathbb{F}^n$ gives a geometrical structure on $\mathbb{F}P^{n-1}$

e.g. $D=$  gives a point on a line in $\mathbb{F}P^2$.

(c) Now, $GL(n,\mathbb{F})$ acts on $\mathbb{F}^n$ hence on $D(\mathbb{F}^n)$ for any uncombed n-box YD

So we get a $GL(n,\mathbb{F})$-representation on $\mathbb{C}^{D(\mathbb{F}^n)}$.

When $\mathbb{F} = \mathbb{F}_q$, alas, not all irreps of $GL(n,\mathbb{F}_q)$ lie in these.

(d) Finally, one can find relations between $GL(n,\mathbb{F})$ and $(n!)$

$GL(n,\mathbb{F})$ acts via symmetries of $\mathbb{F}P^{n-1}$.

(4) $\boxed{\text{Answer to the homework in (3)(c)}}$: We first make the following

(a) $\boxed{\text{Lemma}}$ The number of k-subspaces in $\mathbb{F}_q^n$ (i.e. $(\vec{0},\vec{k},\vec{n})$-flags) is
$$\binom{n}{k}_q := \frac{[n]_q!}{[k]_q!\,[n-k]_q!}$$

(We show this later, but as $q \to 1$ the classical limit of the lemma says that $\#\{\text{subsets of } [n] \text{ of size } k\} = \binom{n}{k}$.)

(b) Assuming the lemma, we first compute the quantum expression. Say $[n_1+n_2+\cdots+n_k = n]$ is D.

Then $\#\, n_1$-subspaces in $\mathbb{F}_q^n$ is $\binom{n}{n_1}_q$

and given such an "n_1"-subspace $\#(\text{"}n_1 \subseteq n_2\text{"}) = \binom{n-n_1}{n_2}_q$

etc.

In other words / Summarizing inductively, the required expression is

$$\frac{[n]_q!}{[n_1]_q!\,[n_2]_q!\cdots[n_k]_q!}$$, the q-multinomial coefficient.

(c) As $q \to 1$, this of course goes to $\binom{n}{n_1,\ldots,n_k}$, the usual multinomial coefficient.

And — surprise, surprise! — so does its "interpretation":

the number of D-flags on the finite set $[n]$, equals

choose n_1 elts. for "last" row
then n_2 more for the next row
...
and so on

$\Rightarrow \dfrac{n!}{n_1!\,n_2!\cdots n_k!}$

$\boxed{\text{NOTE}}$: For this to be the answer, we must have "unordered sets" $X_{i+1} \setminus X_i$ (as we do).

(d) And now for the proof of the Lemma — which concludes this homework.

This is done in a later class.