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(1) (a) Thinking of q as a "phase" gives a quantum Pascal triangle, and every component is

$$\begin{tikzcd} & \bullet \arrow[dl, "1"'] \arrow[dr, "q^k"] & \\ \bullet \arrow[dr, "q^{k+1}"'] & & \bullet \arrow[dl, "1"] \\ & \bullet & \end{tikzcd}$$

i.e. there are two paths, with amplitudes differing by phase q!

This is also manifested in a physical situation: a magnetic field and a charged particle in it. Then two paths around the field give different answers!

(b) So we now think of two processes $\overset{y}{\swarrow}$, $\overset{x}{\searrow}$, and look at

$$\begin{tikzcd} & \bullet \arrow[dl, "y"'] \arrow[dr, "x"] & \\ \bullet \arrow[dr, "x"'] & & \bullet \arrow[dl, "y"] \\ & \bullet \arrow[u, phantom, "q"] & \end{tikzcd}$$

which means $qyx = xy$ (b/c paths are yx & xy)

and then we have the $\boxed{\text{q-binomial theorem}}$ (which can be explained by looking at the sum of weighted paths to any place in the q-Pascal triangle).

$$\boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k}_q y^k x^{n-k}}$$ where $xy = q\,yx$, $q \notin \sqrt{1}$

(2) (a) We're used to polynomial functions (valued in k = field) on the plane k^2:

$$k[x,y] = k\langle x,y\rangle / \langle xy = yx\rangle$$

but now we're thinking about the "quantum plane" (i.e. its "ring of functions"):

This is the noncommutative algebra $k_q[x,y] = k\langle x,y\rangle / \langle xy = qyx\rangle$

And this is the start of a subject called <u>noncommutative geometry</u>.

(b) This has a lot of interesting aspects to it. For example, $GL(2,k)$ acts on k^2 and is the group of symmetries. Similarly the ~~quantum plane (or more~~ <u>quantum group</u> $GL_q(2,k)$ "acts" on the quantum plane (or more precisely, its ring of functions) ($k_q[x,y]$)

(c) <u>More on this</u>: Algebraic geometry is the effort to study spaces by studying functions on them.

Geometry is nice → it lets you visualise!

Algebra is nice → —do— compute!

So we try to get the best of both worlds.

eg. X = space $\Longrightarrow$ Comm alg. $\mathcal{O}(X)$ of fns on X

?? → Can one recover

eg. X = Compact Hausdorff space ⇒ One can recover X from $\mathcal{O}(X)$ → ~~Gelfand~~ Naimark

(the Gelfand-Naimark Theorem)

Sometimes $X = $ set $\Rightarrow \mathcal{O}(X) = $ all fns on X

or $X = $ alg. variety (affine), $\mathcal{O}(X) = $ "regular" maps

(d) Geometry $\longrightarrow$ Algebra

Geometry	Algebra
(i) Space X (say in a category $\mathcal{C}$)	Comm alg $\mathcal{O}(X)$ of fns on X
(ii) Maps $\varphi : X \to Y$ (say morphisms in $\mathcal{C}$)	Algebra morphism $\varphi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$, $\varphi^* = \mathcal{O}(\varphi)$
(iii) Say we take a gp. G: $m : G \times G \to G$; $inv : G \to G$; $id : \mathbb{1} = $ terminal obj. $\to G$	$\mathcal{O}(G) = $ Commutative Hopf alg. $m^* : \mathcal{O}(G) \to \mathcal{O}(G \times G) = \mathcal{O}(G) \otimes \mathcal{O}(G)$; $inv.^* : \mathcal{O}(G) \to \mathcal{O}(G)$; $id^* : \mathcal{O}(G) \to \mathcal{O}(\mathbb{1}) = k$ (initial object in the category of alg's). Called a coalgebra with antipode.

Note: If you start with just a group, you get a Hopf algebra, i.e. one gets an alg. that is also a coalgebra and has an antipode and these structures are all compatible!

(iv) A group G acts on a space X, $\alpha : G \times X \to X$ if ...	~~Algebra~~ The comm Hopf alg. $\mathcal{O}(G)$ "acts" on the comm. alg $\mathcal{O}(X)$: $\alpha^* : \mathcal{O}(X) \to \mathcal{O}(G) \otimes \mathcal{O}(X)$.

(e) The idea, now, is to scratch the word "commutative" out, and hope that the theory still works, and extends to more setups / spaces!

So, $GL_q(2,k)$ is a (non commutative) Hopf algebra, coacting on the quantum plane $k_q[x,y]$!

But so, this seems to be not so motivated, rather, it's a bit formal.

(f) To make non commutative geometry (and more specifically, the q-binomial formula) less formal, let's go back to our chart:

(a)

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\binom{n}{k} \in \mathbb{N} \arrow[r, "q\text{-deforming}"'] \arrow[d, "\text{Categorifying}"'] & \binom{n}{k}_q \in \mathbb{N}[q] \\
\binom{n}{k} \arrow[r, "\mathbb{F}\text{-deformed}"'] & \binom{n}{k}_{\mathbb{F}} = \text{Grassmannian} \arrow[u, "(\mathbb{F}=\mathbb{F}_q)"', "\text{decategorification}"]
\end{tikzcd}$$

we did this via G

~~Grassmannian~~

finite sets

And the question remained: What category does it live in?

We could have said $\to$ Category of alg. varieties $/\mathbb{F}_q$

But we're then interested in breaking up this set into pieces $\to$ and this isn't ~~...~~ in the category of varieties!

(<u>"Guess"</u> $\to$ Motives)

(b) The (q)-binomial formula has / should have a different meaning in each of the four settings.

In the top row, they are both first identities of formal symbols/variables.

What are they in the bottom row? It turns out that in both settings, they are — Hecke operators!

eg: in $\mathbb{F}_q$-, or F- setting, the F-Pascal Δ has

$$\binom{n}{k}_F = \{V \subseteq F^n : \dim_F V = k\}$$

$$\begin{tikzcd}
 & \binom{n}{k}_F \arrow[dl, "y"', "1"] \arrow[dr, "x", "q^{n-k}"'] & \\
\binom{n+1}{k}_F \arrow[dr, "x"', "q^{n+1-k}"] & & \binom{n+1}{k+1} \arrow[dl, "1"', "y"] \\
 & \binom{n+2}{k+1}_F &
\end{tikzcd}
\qquad (*)$$

(c) The y's are easy: Given $V_k \subseteq F^n$,

it uniquely gives $V_k \subseteq F^n \subseteq F^{n+1}$.

So y's have weight 1.

The x's are not as simple: From $\binom{n}{k}_F \longrightarrow$ a $V_{k+1} \subseteq F^{k+1}$

??

There are q^{n-k} ways of doing this!

This gives the relations in $(*)$.

(d) Thus, X, Y are relations ("multi-valued functions") between sets: but they're invariant under the action of $GL(n, F)$
(which acts on all four spaces/sets, via —
Hecke operators again!)

So, we want X, Y to satisfy $XY = qYX$

(f) (a) Next time: We'll see an "isomorphic relation" between

$$\binom{n}{k}_F \xrightarrow{\sim} \binom{n}{n-k}_F$$

which explains this symmetry between q-binomial coefficients — and hence also between q-multinomial coefficients.

(b) Alitea: $V_k \subseteq F^n \longleftrightarrow (V_k)^\perp \subseteq (F^n)^* = F^n$

So decategorifying this proves the symmetry!