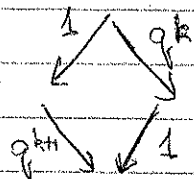


John Baez

① a

Thinking of q as a phase gives a quantum Pascal triangle, and every component is

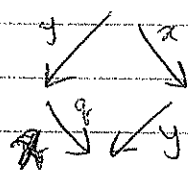


i.e. there are two paths, with amplitudes differing by phase q !

This is also manifested in a physical situation: a magnetic field and a charged particle in it. Then two paths around the field give different answers!

② b

So we now think of two processes $y \swarrow$, $\searrow x$, and look at



which means

$$qyx = xy \text{ (b/c paths are } \swarrow \text{ and } \searrow \text{)}$$

and then we have the q -binomial theorem (which can be explained by looking at the sum of weighted paths to any place in the q -Pascal triangle).

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k}_q y^k x^{n-k} \text{ where } \underline{xy = q, yx}, q \neq \pm 1$$

② a

We're used to polynomial functions (valued in $k = \text{field}$) on the plane k^2 : $k[x,y] = k\langle x,y \rangle / \langle xy = yx \rangle$

but now we're thinking about the "quantum plane", i.e. its "ring of functions":

This is the noncommutative algebra $k_q[x, y] = k\langle x, y \rangle / \langle xy = qyx \rangle$

and this is the start of a subject called noncommutative geometry.

(b) This has a lot of interesting aspects to it. For example, $GL(2, k)$ acts on k^2 and is the group of symmetries. Similarly, the quantum plane (or quantum group $GL_q(2, k)$) "acts" on the quantum plane (or more precisely, its ring of functions) $k_q[x, y]$.

X

(c) More on this: Algebraic geometry is the effort to study spaces by studying functions on them.

Geometry is nice \rightarrow it lets you visualize!

Algebra is nice \rightarrow do — compute!

\Rightarrow we try to get the best of both worlds.

eg. $X = \text{space} \iff$ Comm. alg. $\mathcal{O}(X)$ of fns. on X .
 \uparrow
?? \rightarrow can one recover

eg. $X = \text{Compact Hausdorff space} \Rightarrow$ one can recover X from $\mathcal{O}(X) \rightarrow$ ~~get back~~ Network

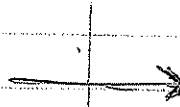
(the Gelfand-Naimark Theorem)

Sometimes $X = \text{set} \Rightarrow \mathcal{O}(X) = \text{all fns on } X$

or $X = \text{alg-variety (affine)}$, $\mathcal{O}(X) = \text{"regular" maps}$

(d)

Geometry



Algebra

(i) Space X (say in a category \mathcal{C})

Comm alg $\mathcal{O}(X)$ of fns on X

(ii) Maps $\varphi: X \rightarrow Y$
(say morphisms in \mathcal{C})

Algebra morphism $\varphi^*: \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$
" $\mathcal{O}(\varphi)$ "

(iii) Say we take a gp: G :
 $m: G \times G \rightarrow G$
 $\text{inv}: G \rightarrow G$
 $\text{id}: \mathbb{1} = \text{terminal obj.} \rightarrow G$

$\mathcal{O}(G) = \text{Commutative Hopf alg.}$

$m^*: \mathcal{O}(G) \rightarrow \mathcal{O}(G \times G) = \mathcal{O}(G) \otimes \mathcal{O}(G)$

$\text{inv}^*: \mathcal{O}(G) \rightarrow \mathcal{O}(G)$

$\text{id}^*: \mathcal{O}(G) \rightarrow \mathcal{O}(\mathbb{1}) = k$

↓ initial object in the category of algs
called a coalgebra with antipode.

[Note: If you start with just a group, you get a Hopf algebra, or one gets an alg. that is also a coalgebra and has an antipode, and these structures are all compatible!]

(iv) A group G acts on a space X
 $\alpha: G \times X \rightarrow X$ if...

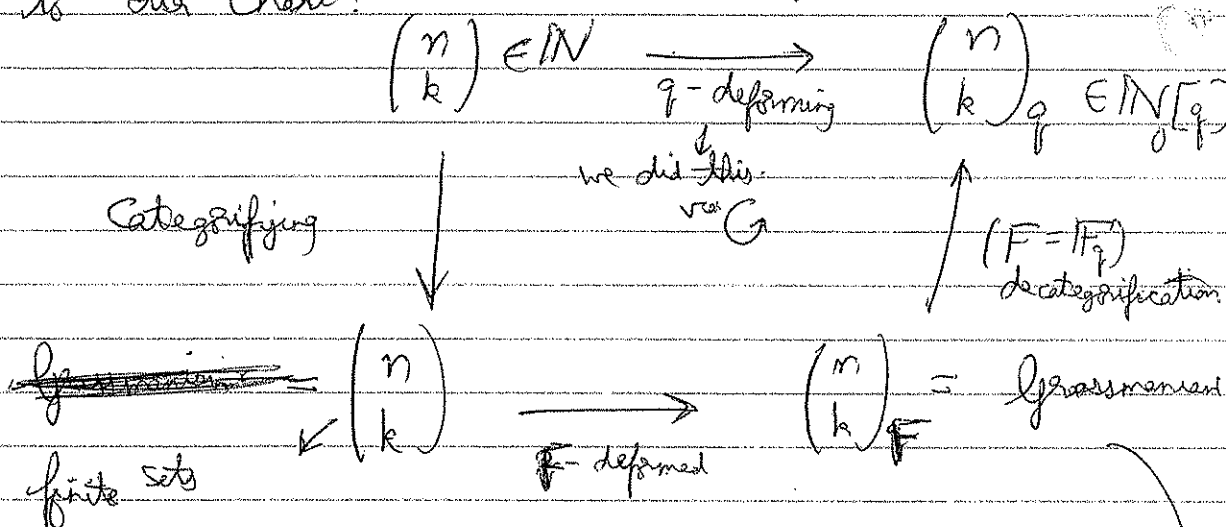
~~Again~~ The comm Hopf alg. $\mathcal{O}(G)$ "acts" on the comm. alg $\mathcal{O}(X)$: $\alpha^*: \mathcal{O}(X) \rightarrow \mathcal{O}(G) \otimes \mathcal{O}(X)$

e) The idea, now, is to scratch the word "commutative" out, and hope that the theory still works, and extends to more setups / spaces!

So, $GL_q(2, k)$ is a (non commutative) Hopf algebra, coacting on the quantum plane $k_q[x, y]$.

But so, this seems to be not so motivated, rather, it's a bit formal.

To make non commutative geometry (and more specifically, the q -binomial formula) less formal, let's go back to our charts:



And the question remained: What category does $\binom{n}{k}_F$ live in?

We could have said \rightarrow Category of alg. varieties / \mathbb{F}_q

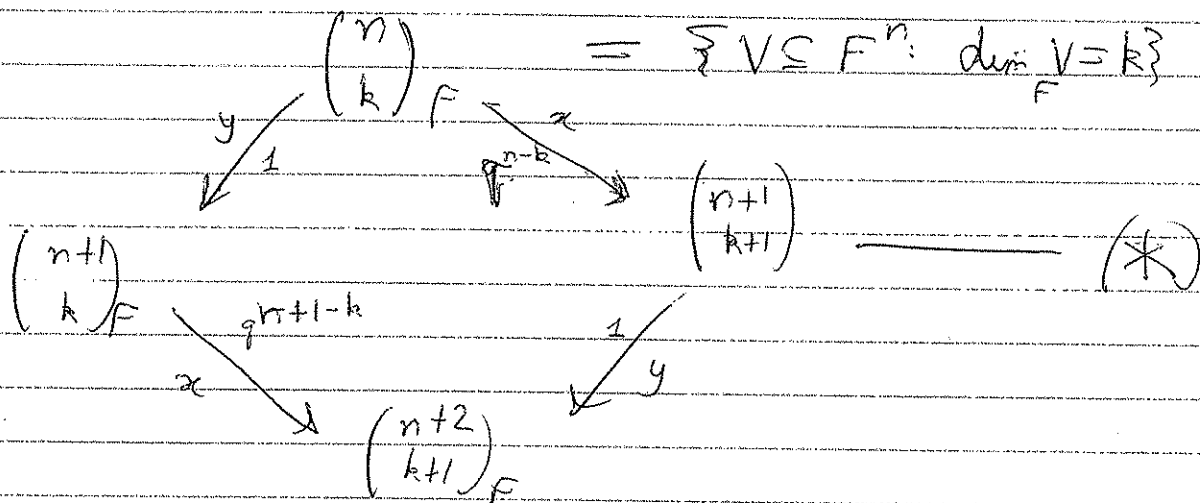
But we're then interested in breaking up this set into pieces \rightarrow and this isn't ~~in~~ in the category of varieties! ("Guess" \rightarrow Motives)

(b) The (q) -binomial formula has / should have a different meaning in each of the four settings.

In the top row, they are both just identities of formal symbols/variables.

What are they in the bottom row? It turns out that in both settings, they are — Hecke operators!

eg: in q - & F - setting, the F -Pascal Δ has



(c)

The y 's are easy: Given $V_k \subseteq F^n$,

it uniquely gives $V_k \subseteq F^n \subseteq F^{n+1}$.

So y 's have weight 1.

The α 's are not so simple: From $\binom{n}{k}_F \rightarrow a V_{k+1} \subseteq F^{k+1}$

There are F^{n-k} ways of doing this! \downarrow ??

This gives the relations in (*).

(d)

Thus, X, Y are relations ("multi-valued functions")

between sets; but they're invariant under the action of

$GL(n, F)$

(which acts on all four spaces/sets, via —

Hecke operators again!)

So we want

X, Y to satisfy $XY = qYX$

(e) (a)

Next time: We'll see an "isomorphic relation" between

$$\binom{n}{k}_F \xrightarrow{\sim} \binom{n}{n-k}_F$$

which explains this symmetry between q -binomial coefficients — and hence also between q -multinomial coefficients.

(b)

Aliter: $V_R \subseteq F^{n \times 1} \iff (V_R)^\perp \subseteq (F^n)^* = F^n$

So decategorifying this proves the symmetry!