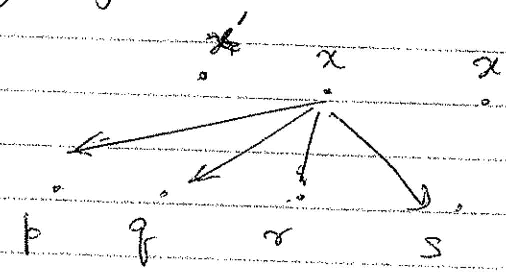
NOV 6 20..

John Baez : Matrix Mechanics

(1) Say X, Y, Z are (finite) sets.

(a) Heisenberg was interested in physical processes that take one physical system (i.e. a set of states) to another.

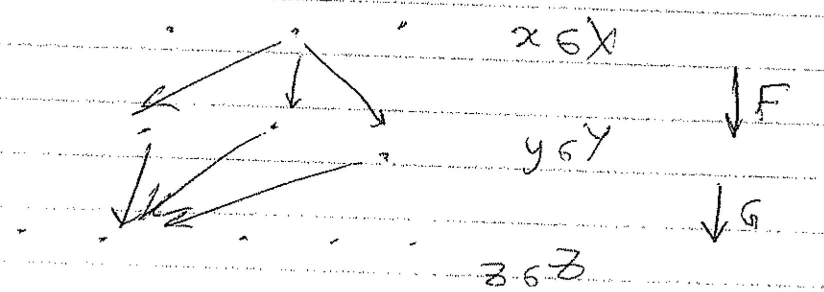
Then maybe one doesn't know *definitely* which state goes to which one → So we write down an *amplitude* for each possibility. eg



~~A_{xp}, A_{xq}, etc.~~

$$F_{xp},\ F_{xq},\ F_{xr},\ F_{xs} \in \mathbb{C}$$

(b) What about composing two processes?



Then (Heis —) $(G \circ F)_{xz} = \sum_{y \in Y} G_{yz} F_{xy}$ (all entries are $\in \mathbb{C}$)

$$= \sum_{y \in Y} F_{xy} G_{yz}$$

So this really is (the transpose of) matrix mult!

then we deal with Hilbert ~~space~~ spaces
$\mathbb{C}^X$, & ~~~~ elements are just wave-functions

(c) This means that we're doing $F: X \times Y \to \mathbb{C}$ etc.

Moreover, we really are only using $(\mathbb{C}, +, \cdot)$, not $-$ or $/$

So, we're not using the field axioms, just the rig axioms!

(d) Let's do matrix mechanics with other rigs — e.g. the Boolean rig $\{0,1\} = \{F,T\}$ replaces $\mathbb{C}$ now.

Then $F: X \to Y$ is given by its possibility matrix (each edge stands for a "T")!

$+ \to$ OR
$\cdot \to$ AND } and now the ("graph of") F is just that of a relation!

$F: X \times Y \to \{0,1\}$

or "linear" op. $F: \{0,1\}^X \to \{0,1\}^Y$.

and $\{0,1\}^X$ are also called subsets of X!

(e) One can bring back $\mathbb{C}$ — and Hecke operators! — from relns:

Given a relation $F: X \times Y \to \{0,1\}$, you can interpret it as a linear operator $\tilde{F}: X \times Y \to \mathbb{C}$

using $i: \{0,1\} \hookrightarrow \mathbb{C}$.

If G acts on X & Y, and F is G-equivariant then G acts on the perm. rep.'s $\mathbb{C}^X$, $\mathbb{C}^Y$, and

The Hecke operators

then $\tilde{F} : X \times Y \to \mathbb{C} \Rightarrow \tilde{F} : \mathbb{C}^X \to \mathbb{C}^Y$ is also G-equivariant.

(f) The only problem in $F \to \tilde{F}$ is: is $\tilde{F_1}\tilde{F_2} = \widetilde{F_1 F_2}$?

Because we do know that $\tilde{1} = 1$ $(1 : X \times X \to \mathbb{C} = \delta_{xx'})$

So if $\tilde{F_1}\tilde{F_2} = \widetilde{F_1 F_2}$, then we could say that this is

$\Leftrightarrow$ $\sim : \underset{\boxed{\text{FIN-REL}}}{\begin{pmatrix}\text{Relations between} \\ \text{finite sets}\end{pmatrix}} \longrightarrow \begin{pmatrix}\text{Operators on fin.} \\ \text{dim. vector spaces}\end{pmatrix} = \boxed{\text{FIN-VECT}}$

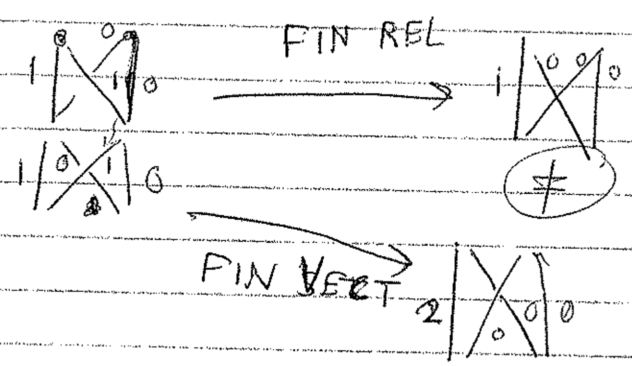
is a functor.

So, is it? <u>NO</u>, because $v : \{0,1\} \hookrightarrow \mathbb{C}$ is <u>NOT</u> a ring ~~homo~~ homomorphism.

It preserves AND & $\cdot$, but not OR $\to$ +

1 OR 1 $\neq$ 1 ☹ !

(g) $\boxed{\text{example}}$



(h) So how does one correct this? Instead of $\{0,1\}$-valued matrices, we could use $\mathbb{N}_0$-valued matrices, since

ii) $\mathbb{N} \hookrightarrow \mathbb{C}$ is a rig homomorphism

This is related to $\{0,1\}$-valued matrices, via the rig homomorphism $\mathbb{N}_0 \twoheadrightarrow \{0,1\}$

$$\begin{pmatrix} n>0 \to 1 \\ 0 \to 0 \end{pmatrix}$$

We could also categorify this — then we use rig-categories instead of rigs! eg. FIN-SET-valued matrices, also called spans!

$+ \longrightarrow$ coproduct $=$ disjoint union $\coprod$
$\times \longrightarrow$ product $=$ Cartesian product.

(c) NOTE: FIN-SET is not a rig, because it's not a set!

But modulo natural transformations (which quotients out FINSET so that we end up with a set $\mathbb{N}_0$),

$$\text{FIN SET} \xrightarrow[\text{decat.} = |\cdot|]{} \mathbb{N}_0 \to \mathbb{C} \quad \text{is a "rig map".}$$

So the GoF-picture from (b) above, would now change into

$$(GF)_{ok} = \coprod_{j \in Y} G_{jk} \times F_{ij}$$

and then $|\widetilde{G \cdot F}| = |\widetilde{F}||\widetilde{G}|$

(2) (a) A matrix ~~F~~ $F : X \times Y \to \mathrm{FINSET}$ is also called a <u>span</u>, since we define

$$S = \coprod_{\substack{i \in X \\ j \in Y}} F_{ij}$$

Define

$$\begin{tikzcd} S \arrow[r, "p_1"] \arrow[dr, "p_2"'] & X \\ & Y \end{tikzcd}$$

via:

$\forall s \in S, \exists! (i,j) : s \in F_{ij}$. Define $p_1(s) = i$, $p_2(s) = j$.

In other words, $S = \{F_{ij}\} = \{\text{arrows} : X \to Y\}$, and

$p_1 =$ source map : arrows $\to X$
$p_2 =$ target map : arrows $\to Y$.

This is good / has all the information, since, eg

$$F_{ij} = \{ s \in S : p_1(s) = i,\ p_2(s) = j \}$$

(b) How, in this notation, does matrix multiplication look like?

$$\begin{tikzcd} & & ST \arrow[dl, dashed] \arrow[dr, dashed] \arrow[ddll, bend right] \arrow[ddrr, bend left] & & \\ & S \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & T \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ X & & Y & & Z \end{tikzcd}$$

In fact, $ST = \{ (s,t) : s \in S, t \in T, p_2(s) = q_1(t) \}$

So this is the <u>pullback</u> of

$$\begin{tikzcd} S \arrow[dr] & & T \arrow[dl] \\ & Y & \end{tikzcd}$$

(c) This makes Spans seem like morphisms in some category whose objects ~~the~~ are finite sets, since we can now *compose*!

~~So~~ let us think of all the data that we *do* have — and what we need, to make it a category.

- FIN-SET = finite sets for objects
- Spans of finite sets: $S : X \nrightarrow Y$, ie

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

- Composition via pullbacks
- identity "morphisms"

$$\begin{tikzcd} & X \arrow[dl, "1_X"'] \arrow[dr, "1_X"] & \\ X & & X \end{tikzcd}$$

What about *associativity*? NOT equal, but isomorphic!

$$(ST)U \cong S(TU)$$

$$\begin{tikzcd}
& S(TU) \arrow[rrrr, leftrightarrow, "\cong"] \arrow[dd] \arrow[drrr] & & & & (ST)U \arrow[dlll] \arrow[dd] & \\
& & ST \arrow[dl] \arrow[dr] & & TU \arrow[dl] \arrow[dr] & & \\
& S \arrow[dl] \arrow[dr] & & T \arrow[dl] \arrow[dr] & & U \arrow[dl] \arrow[dr] & \\
W & & X & & Y & & Z
\end{tikzcd}$$

This is essentially because pullbacks are not equal but $\cong$.

In general, if we're forming something in two different ways, and these ways use some universal objects, we should expect both outputs not necessarily to be equal, but isomorphic!

(d) So there really is a <u>bicategory</u> here, of

- finite sets as objects
- spans of finite sets
- isomorphisms of spans

Equivalently, there's a category FIN-SPAN, of

- fin sets as objects
- isomorphism classes of spans as morphisms

And we're now getting a functor : FIN-SPAN $\to$ FIN-VECT
So this is a good thing, given our initial goal!

(e) <u>Isoclasses of spans</u>: Morphisms of spans should be

$$\begin{tikzcd} S \arrow[r] \arrow[d] \arrow[dr] & T \arrow[dl] \arrow[d] \\ X & Y \end{tikzcd}$$

So we'll see next time onwards ~~that~~ this approach (via spans) to ~~derived categories~~ Hecke operators, is the (/a more) "correct" way to go.