8 | NOV | 2007 | THU

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(1) Last time we saw how to "compose" or "expected values in composing" Hecke operators.

Today we start by contrasting that construction, to ordinary matrix multiplication.

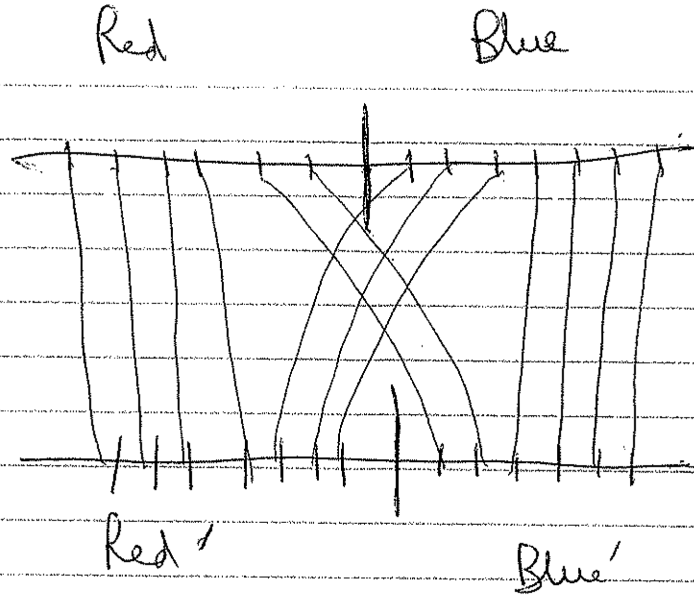
"Ordinary"	"Weird"
We make an algebra, where matrices are elements.	We make an algebra where the matrices are basis elements.
Thus, matrix mult. is "well-def": it takes matrices to matrices.	Here, multiplying two of these doesn't give a single matrix, but a linear combination. In very vague terms, mult. here is "fuzzy". (Or, we get, instead of a binary operation, a "bilinear operation".)

(2) (a) We now ~~[illegible]~~ introduce another ~~[illegible]~~ system of notation which still represents relationships between different flags, and is to be suggestive of something else

We do this via this matrix notation.

$$\begin{array}{c|cc} & R=6 & 7=B \\ \hline R'=7 & 4 & 3 \\ B'=6 & 2 & 4 \end{array}$$

So we now introduce, for this matrix, the braid notation, as follows:

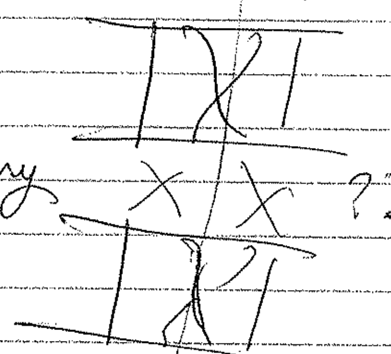


and this works for any $n \times n$ matrix with coefficients in $\mathbb{N}$.

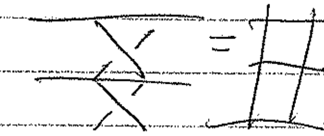
(b) This is suggestive in the sense that ~~there~~ it makes us think of permutations (eg $\cong 13!$).

Also suggestive in the manner of composing Hecke operators → as in the last lecture, one doesn't just compose, but allows to permute in between!

This is what can give many different outputs, as in last class



(c) Moreover, in braids, we don't have →



~~but~~ which makes sense as far as permutations go, because

$S_n = Br_n / (\sigma_i^2 = 1 \ \forall\, 0 \leq i < n)$. So if we only care about perm's, then the 2-dim'al projections of the braids will work.

(d) However, when does the "3-D" part of the information actually say something useful?

$\boxed{\text{ANS}}$ When there's a "q" floating around!

Here's some explanation / observations:

13! is supposed to look like $GL(13, \mathbb{F}_1)$

because ~~it~~ ~~also~~ is related to $GL(13, \mathbb{F}_q)$.

These two are related because the binary relationships between flags in $\mathbb{F}_q^{13}$ and $\mathbb{F}_1^{13}$ are the same!

(e) But ~~then~~, we really should ask also – Is 13! the $q \to 1$ limit of some actual, concrete algebra?

ANS: YES! This is the Hecke algebra $\mathcal{H}_n$ of 13!

(see next page)

(f) Moreover, this Hecke algebra is related to the non-commuting quantum n-space $_{(n=13)}$ ($X_i X_j = q X_j X_i$?), just as in the case $q \to 1$, 13! relates to $k[X_1, \dots, X_{13}]$.

(3) Postscript (by John Baez) :

Given any field k, we have $GL_q(n,k)$ = invertible $n \times n$ matrices with coeff's in k.

This acts on its "fundamental representation" $k^n = V$

Moreover, this will mean that $GL(n,k)$ acts on $\mathrm{Sym}\, V$

e.g. $GL(2,k)$ acts on $\mathrm{Sym}(k^2) = k[X,Y]$

And for any fixed N, we have $GL(2,k)$ acts on

$k[X,Y]^{\otimes n}$ $\circlearrowleft$ S_n acts via permuting!

This is some sort of Schur–Weyl theory, and remarkably, it also can be q-deformed meaningfully!

$GL_q(2,k)$ = "quantum GL_2" acts on $k_q[X,Y]$ = quantum plane

$= k\langle X,Y\rangle / (XY - qYX)$

and on $k_q[X,Y]^{\otimes n}$, we look at not the S_n-action

(for something "Schur-Weyl-compatible" with GL_q),

but at the Hecke algebra $\mathcal{H}_n(q)$, acting on it!

And then there is some Schur–Weyl analogue

(In some sense, S_n is not the right thing for non-cocomm of GL_q?)