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John Baez (More on Matrix Mechanics)

⓪ (Also note, from last time, ~~that~~ the "braid picture" is related to Hecke algebras, because H_n is also a quotient of the braid group B_n, just as S_n is!)

① We saw last week that for any rig R, there's a category

ⓐ $\mathrm{Mat}(R)$, where

- objects = finite sets $X, Y, \dots$
- morphisms = $f : X \nrightarrow Y$ is a matrix, i.e. $f : X \times Y \to R$
- Composition is matrix multiplication

ⓑ Given a rig homomorphism $\varphi : R \to S$, we get a functor

$\mathrm{Mat}(\varphi) = \varphi_* : \mathrm{Mat}(R) \to \mathrm{Mat}(S)$

(So Mat itself is a functor: $\mathrm{RIG} \to \mathrm{CAT}$!)

ⓒ For example, a relation $: X \to Y$ is a morphism in $\mathrm{Mat}(\mathrm{BOO})$ where BOO = Boolean Rig $= (\{0,1\}, \mathrm{AND}, \mathrm{OR})$

The idea was to change such a relation into a Hecke operator $\in \mathrm{Mat}(\mathbb{C})$. But the problem was that the inclusion $: \{0,1\} \hookrightarrow \mathbb{C}$ is not a rig homomorphism!

This was rectified using the rig $\mathbb{Z}_{\geq 0} = \mathbb{N}_0$ $\begin{smallmatrix} \nearrow \{0,1\} \\ \searrow \mathbb{C} \end{smallmatrix}$

(2) Moreover, $\mathbb{N}$ is ~~the~~ free rig on one generator, and the initial "object" in the category RIGV ~~which is why~~
$\left(\begin{smallmatrix} 0 \to 0 \\ 1 \to 1 \end{smallmatrix} \Rightarrow \checkmark\right)$

(d) This explains the previous picture. Now apply Mat:

$$\begin{tikzcd}
 & & \mathrm{Mat}(\{0,1\}) \simeq \mathrm{FinRel} \arrow[d, dashed, "\text{not a functor!}"] \\
\mathrm{FinSpan} & \mathrm{Mat}(\mathbb{N}_0) \arrow[l, "\simeq"'] \arrow[ur] \arrow[r] \arrow[d] & \mathrm{Mat}(\mathbb{C}) \simeq \mathrm{FinVect}_{\mathbb{C}} \\
 & \left(\begin{smallmatrix}\text{objects} \\ = \text{fin sets}\end{smallmatrix}\right) & X \longmapsto \mathbb{C}^X
\end{tikzcd}$$

There are <u>two drawbacks</u>:

(JB) (a) $\downarrow$ is not a functor

(JD) (b) If a group acts on FinRel, then atomic invariant relations do not compose to just one ~~do~~ relation!

(3) Recall that FinSpan is the category coming from a 2-category with

- finite sets as objects
- spans $\begin{tikzcd}[column sep=small] & S \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$ as 1-morphisms $f: X \nrightarrow Y$
- $\begin{tikzcd} S \arrow[r] \arrow[d] \arrow[dr] & S' \arrow[dl] \arrow[d] \\ X & Y \end{tikzcd}$ (commuting) as 2-morphisms.

by decategorifying it – which yields

- fin sets as objects
- isoclasses of spans as morphisms.

Then $(\mathrm{FinSpan})^{\mathrm{decat}} \cong \mathrm{Mat}(\mathbb{N}_0)$

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(2) (a) We have a functor : Fin Span $\longrightarrow$ Fin $\mathrm{Vect}_{\mathbb{C}}$

and for a fin group G acting on everything in sight,

$F : G\text{-FinSpan} \longrightarrow \mathrm{Fin\,Rep}(G)_{\mathbb{C}}$

objects are finite dim G-reps
morphisms are intertwiners

objects are finite G-sets
morphisms are spans of objects

$\boxed{\text{Defn}}$ Morphisms in the image of F are called <u>Hecke operators</u>.

(earlier, they were just morphisms in $\mathrm{Mat}(\{0,1\})$ ...)

One part of

(b) the main theorem of this class ($F(X) = \mathrm{Free}(X)$!) talks about the "fullness" of F.

~~Theorem Given~~

$\boxed{\text{Theorem}}$ Given finite G-sets X & Y, every morphism from $F(X)$ to $F(Y)$ is a $\mathbb{C}$-linear combination of Hecke operators.

However there are redundancies, eg if $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ = Hecke Operator

then so is $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$.

So we need a "better" formulation.

(c) In $\text{FinRep}(G)_{\mathbb{C}}$ hom-sets are free $\mathbb{C}$-modules.
In G-FinSpan, hom-sets are free $\mathbb{N}_0$-modules

$|\text{basis}| = \# \text{orbits of } X \times Y$

Now our main result says

$$\boxed{\hom(F(X), F(Y)) \cong \hom(X,Y) \otimes_{\mathbb{N}_0} \mathbb{C}}$$

(tensoring can be extended to over rigs)

(d) To state this more precisely, we say that

$\text{FinRep}(G)_{\mathbb{C}}$ is enriched over $\mathbb{C}$-Vect,
G-FinSpan — do — Free($\mathbb{N}_0$-mod)
and we have a base change functor: $\mathbb{C} \otimes_{\mathbb{N}_0} -$

: Free($\mathbb{N}_0$-mod) $\to$ $\mathbb{C}$-Vect

that goes in between them.

(e) This construction should hold if we replace $\mathbb{C}$ by any field k, because there still is a rig hom $\mathbb{N}_0 \to k$. Then the following should hold:

$\boxed{\text{Theorem}}$ For any field k, $F: G\text{-FinSpan} \to \text{FinRep}(G)_k$, and $F: \hom(X,Y) \to \hom(F(X), F(Y))$ gives an isom in k-Vect, via:

$$\hom(F(X), F(Y)) \cong \hom(X,Y) \otimes_{\mathbb{N}_0} k.$$

(For all we know, it may hold for k a unital commutative ring!)

(3) <u>(Back to what we were doing)</u>

We were considering certain specific spans

(a) $$X : \binom{n}{k}_F \nrightarrow \binom{n+1}{k}_F$$

$$Y : \binom{n}{k}_F \nrightarrow \binom{n+1}{k+1}_F \quad \text{for any field } F.$$

Here, X takes ~~k-space in n-space~~ a k-space in n-space to a k-space in $(n+1)$-space.

This is only doable in one way; thus this span is a <u>function</u>!

$$\begin{tikzcd} & \bullet \arrow[dl] \arrow[dr] \arrow[drr, bend left] & & \\ \binom{n}{k}_F & & \binom{n+1}{k}_F & \end{tikzcd}$$

This is $\binom{n}{k}_F$! So, the span is

$$\begin{tikzcd} & V \subseteq F^n \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ V & & V \subseteq F^n \times F = F^{n+1}. \end{tikzcd}$$

Y is more complicated:

$$\begin{tikzcd} & ? \arrow[dl] \arrow[dr] & \\ \binom{n}{k}_F & & \binom{n+1}{k+1}_F \end{tikzcd}$$

This is for next time!