

13/NOV/2007/TUE

John Baez (More on Matrix Mechanics)

① (Also note, from last time, that the "braid picture" is related to Hecke algebras, because H_n is also a quotient of the braid group B_n , just as S_n is!)

① We saw last week that for any rig R , there's a category

② $\text{Mat}(R)$, where

- objects = finite sets X, Y, \dots
- morphisms = $f: X \rightarrow Y$ is a matrix, i.e. $f: X \times Y \rightarrow R$

• Composition is matrix multiplication

③ Given a rig homomorphism $\varphi: R \rightarrow S$, we get a functor

$$\text{Mat}(\varphi) = \varphi_* : \text{Mat}(R) \rightarrow \text{Mat}(S)$$

(So Mat itself is a functor: $\text{RIG} \rightarrow \text{CAT}$!)

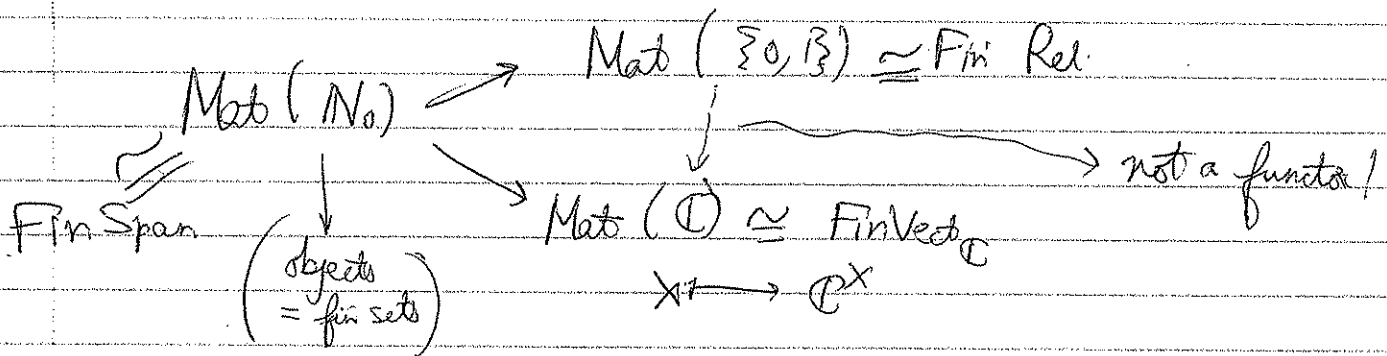
④ For example, a relation $X \rightarrow Y$ is a morphism in $\text{Mat}(\text{BOO})$ where $\text{BOO} = \text{Boolean Rig} = (\{0, 1\}, \text{AND}, \text{OR})$.

The idea was to change such a relation into a Hecke operator $\in \text{Mat}(\mathbb{C})$. But the problem was that the inclusion $\{0, 1\} \hookrightarrow \mathbb{C}$ is not a rig homomorphism!

This was rectified using the rig $\mathbb{Z}_{\geq 0} = \mathbb{N}_0 \begin{matrix} \nearrow \{0, 1\} \\ \searrow \mathbb{1} \end{matrix}$

Moreover, \mathbb{N} is the free alg on one generator, and the initial object in the category $\text{RIG} \left(\begin{matrix} 0 \rightarrow 0 \\ 1 \rightarrow 1 \end{matrix} \right)$ ~~with~~ ~~to~~ ~~why~~

(d) This explains the previous picture. Now apply Mat :

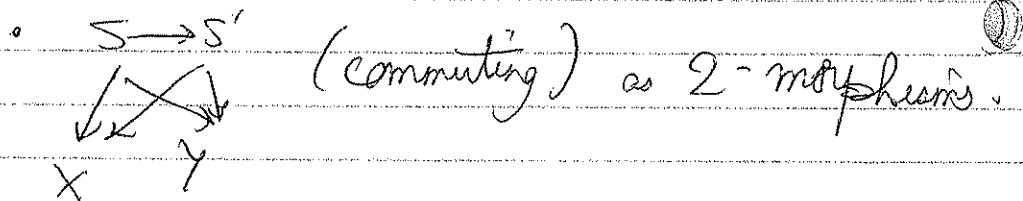
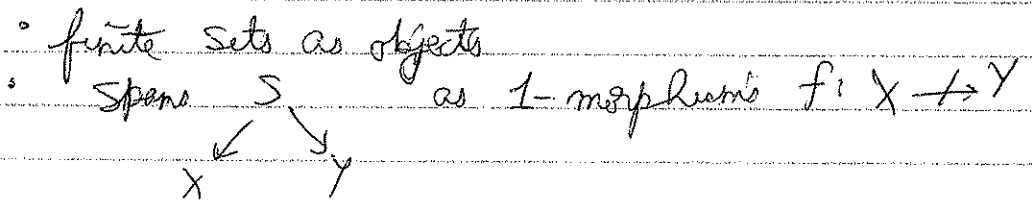


There are two drawbacks:

(JB) (a) \downarrow is not a functor

(JD) (b) If a group acts on FinRel , then atomic ~~invariant~~ relations do not compose to just one relation!

Recall that FinSpan is the ~~category~~ category coming from a 2-category with



by decategorifying it - which yields

- fin sets as objects
- isos of spans as morphisms.

$$\text{Then } (\text{Fin Span})^{\text{decat}} \cong \text{Mat}(\mathbb{N}_0)$$

② @ We have a functor $\text{Fin Span} \rightarrow \text{Fin Vect}_{\mathbb{C}}$

and for a fin group G acting on everything in sight,

$$F: G\text{-Fin Span} \rightarrow \text{Fin Rep}(G)_{\mathbb{C}}$$

objects are finite G -sets
morphisms are spans of objects

objects are finite dim G -reps
morphisms are intertwiners

Defn Morphisms in the image of F are called Hecke operators.

(earlier, they were just morphisms in $\text{Mat}(\mathbb{C}, \mathbb{C}) \dots$)

One part of

⑤ the main theorem of this class is $(F(X) = \text{Free}(X) !)$
talks about the "fullness" of F .

~~Theorem~~ Given

Theorem Given finite G -sets X & Y , every morphism from $F(X)$ to $F(Y)$ is a \mathbb{C} -linear combination of Hecke operators.

However, these are redundancies, eg. of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \text{Hcke operator}$

then so is $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$.

So we need a "better" formulation.

- ⓐ In $\text{Fin Rep}(G)_{\mathbb{C}}$ hom-sets are free \mathbb{C} -modules.
In $G\text{-Fin Span}$ hom-sets are free \mathbb{N}_0 -modules.

$$|\text{basis}| = \# \text{ orbits of } X \times Y$$

Now our main result says

$$\text{hom}(F(X), F(Y)) \cong \text{hom}(X, Y) \otimes_{\mathbb{N}_0} \mathbb{C}$$

(tensoring can be extended to over rigs)

- ⓓ To state this more precisely, we say that

$\text{Fin Rep}(G)_{\mathbb{C}}$ is enriched over $\mathbb{C}\text{-Vect}$,
 $G\text{-Fin Span}$ — do — $\text{Free}(\mathbb{N}_0\text{-mod})$
and we have a base change functor: $\mathbb{C} \otimes_{\mathbb{N}_0} \text{Free}(\mathbb{N}_0\text{-mod}) \rightarrow \mathbb{C}\text{-Vect}$

that goes in between them.

- ⓔ This construction should hold if we replace \mathbb{C} by any field k ,

because there still is a ring hom $\mathbb{N}_0 \rightarrow k$. Then the following should hold:

Theorem For any field k , $F: G\text{-FinSpan} \rightarrow \text{Fin Rep}(G)_k$,
and $F: \text{hom}(X, Y) \rightarrow \text{hom}(F(X), F(Y))$ gives an isom
in $k\text{-Vect}$, ~~in~~ via

$$\text{hom}(F(X), F(Y)) \cong \text{hom}(X, Y) \otimes_{\mathbb{N}_0} k.$$

(For all we know, it may hold for k a unital commutative ring!)

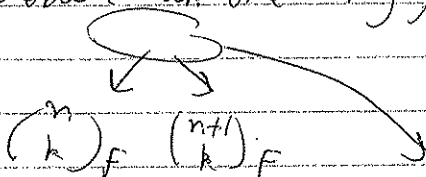
③ (Back to what we were doing)

We were considering certain specific spans

① $X: \binom{n}{k}_F \rightarrow \binom{n+1}{k}_F$
 $Y: \binom{n}{k}_F \rightarrow \binom{n+1}{k+1}_F$ for any field F .

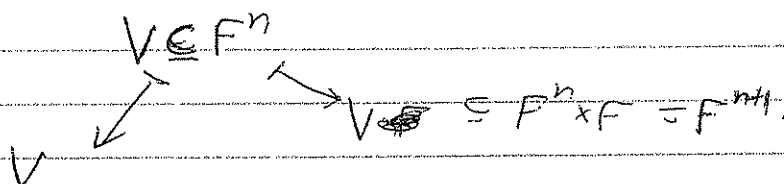
Here, X takes ~~space in space~~ a k -space in n -space to a k -space in $(n+1)$ -space.

This is only doable in one way; thus this span is a function.

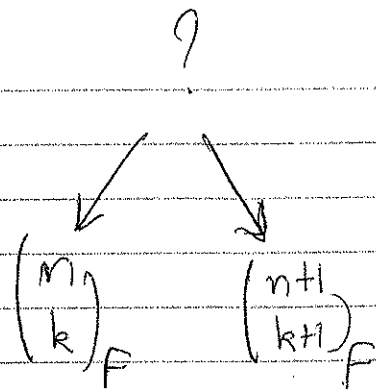


This is

$\binom{n}{k}_F$ | so, the span is



γ is more complicated!



This is for next time!