15 / NOV / 2007 / THU

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(1) What JB said last time actually is not true!

Namely,

(a)

$$\begin{tikzcd}
\underset{\text{Iso classes of spans}}{\underline{G\text{-Set}}} \arrow[r] \arrow[drr, bend right, "F"'] & \underset{\text{Hecke operators}}{\underline{G\text{-set}}} = \underline{\text{Perm}^n\ G\text{-rep}} \arrow[dr, hook] & \\
& & \underline{G\text{-Rep}}
\end{tikzcd}$$

and we claimed that $\hom(F(X), F(Y)) \cong k \otimes_{\mathbb{N}_0} \hom(X, Y)$

(b) But actually, this is false at a very fundamental level!

Because a span in the category of G-sets is not just a set

$$\begin{tikzcd}
& \coprod \{F_{ij}\} \arrow[dl] \arrow[dr] & \\
X & & Y
\end{tikzcd}$$

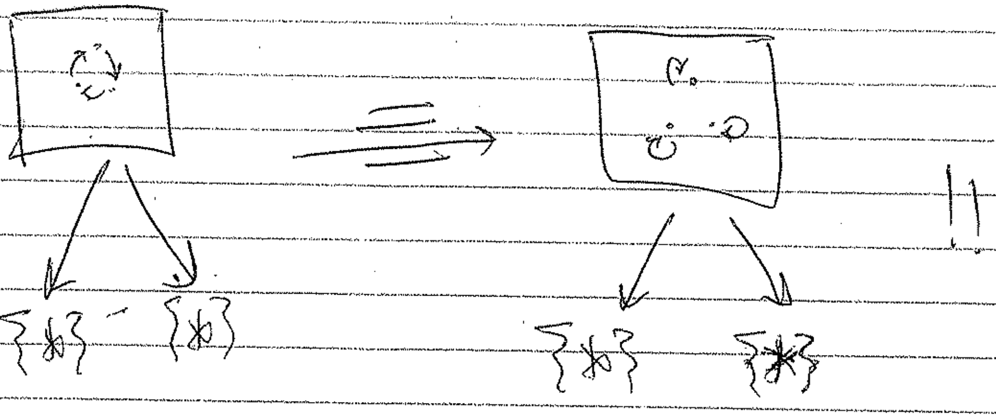
but also a G-action on the top set s.t. $\swarrow, \searrow$ are maps of G-sets!

Now, we said that if $\forall i,j$ $|F_{ij}| = |F'_{ij}|$ then we get the same Hecke operator.

This already kills / identifies any ~~two~~ G-actions on each F_{ij}! e.g. Say $|X| = |Y| = 1$.

$$\begin{tikzcd}
& S \arrow[dl] \arrow[dr] & \\
\{*\} & & \{*\}
\end{tikzcd}$$

and then all maps $\swarrow, \searrow$ are maps of G-sets. But then we're identifying



And this is bad, because it means ~~that~~

→ yes, $\hom(X,Y)$ is a free $\mathbb{N}_0$-module

→ No, the map $k \otimes_{\mathbb{N}_0} \hom(\ )$ is not 1-1; it has a huge kernel!

(c) So, essentially, JB used the wrong decategorification process:

$$
\begin{tikzcd}
\mathrm{Span}(A,B) = \text{Category} \arrow[rr] \arrow[dr] & & V \arrow[dd] \\
& \mathrm{Span}_{\text{isoclasses}}(A,B) = \text{Set} \arrow[ur, "\otimes_{\mathbb{N}_0} k"] \arrow[dr] & \\
& & \mathrm{Hom}(\mathrm{Free}(A), \mathrm{Free}(B))
\end{tikzcd}
$$

—X—

(2) We now ~~and~~ try and understand better the processes of categorification and decategorification.

Cat^n is hard and ambiguous; decat^n is "well-defined"

Examples: (a) ~~Sm~~ Small Cat $\xrightarrow[\text{Iso classes}]{}$ Set

e.g. (b)

$$\begin{array}{ccc} \text{FinSet} & \xrightarrow[\text{"Cardinality"}]{} & \mathbb{N}_0 \\ \text{Free} \downarrow & G & \downarrow \text{id} \\ \text{FinVec}_k & \xrightarrow[\text{"dim"}]{} & \mathbb{N}_0 \end{array}$$

($\rightarrow$ = decat^n, $\downarrow$ = functors)

[(c) Fin G-Set $\xrightarrow[\#\text{orbits}]{}$ $\mathbb{N}_0$] = very small/specific example since it involves a specific gp G!

(d) Small Cat $\xrightarrow[\text{Components}]{}$ Set

(Component = symm. closure of $A \to B$ if $\mathrm{Mor}(A,B) \neq \emptyset$)

~~(e) X = top space $\xrightarrow[\text{path connected}]{}$ Comp.~~

(e)

$$\begin{array}{ccc} \text{TOP. SPACES} & \xrightarrow[\pi_0]{} & \text{SET} \\ \text{id} \downarrow & & \downarrow \pi_{-1} \\ \text{TOP SPACES} & \xrightarrow{\pi_{-1}} & \text{TRUTH VALUES} \end{array}$$

SET = path connected comp.

$\emptyset \to 0$

$(X \neq \emptyset) \to 1$

e.g. A Sol^n is — / All sol^{ns} are {—}

to be replaced by "$\exists$ solution"

(f) $\begin{pmatrix}\text{Abelian}\\ \text{Cat}\end{pmatrix} \xrightarrow{K_0} \begin{pmatrix}\text{Abelian}\\ \text{Group}\end{pmatrix}$ This is "more destructive" than (iso classes)

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(3) (a) Compared to this is the idea that we have to be more destructive than what JB did last time

This is the suggested correction:

$$\underline{\text{Groupoid}}_{\underline{\text{Span}}} \xrightarrow{\text{"degroupoidification"}} \underline{\text{Vect}}_k$$

(b) All ~~things~~ decategorifications above are going from n-cat $\to (n-1)$-cat for some n.

e.g. $\underline{\text{Cat}}\ (= 2\text{-Cat}) \to \underline{\text{Set}}$
$= (1\text{-Cat})$

for ~~ex~~ if we want to consider a 2-cat structure 2-morphisms ~~are~~ only the identity!

But this is a pathological example!