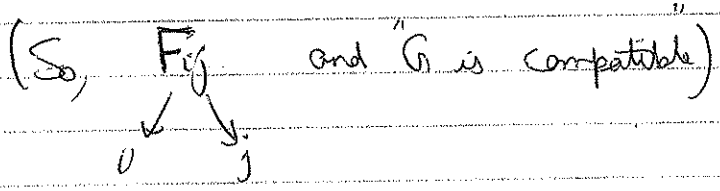
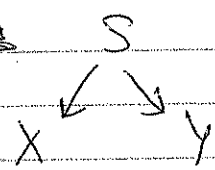


20/NOV/2007/TUE

John Baer

① Theorem (k any field) Let G be a finite group, and X, Y be finite G-sets. Let $\text{hom}(X, Y)$ be the set of isomorphisms of spans of G-sets



This is a finitely generated \mathbb{N}_0 -module, and we have a map

$$\text{hom}(X, Y) \rightarrow \text{Hom}_G(k^X, k^Y) = \{ \text{intertwining operators } f: k^X \rightarrow k^Y \}$$

for any field k. The resulting map

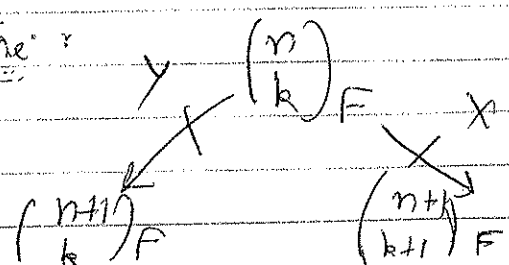
$$1: \text{hom}(X, Y) \otimes_{\mathbb{N}} k \rightarrow \text{Hom}_G(k^X, k^Y)$$

is onto, but not necessarily 1-1.

② To state a better theorem, the "correct" decategorification needs to be used: - not to come to the 1-category of isomorphisms of G-spans, but to use spans of groupoids - which we'll see later in this course.

We'll ~~to~~ continue from where we left off last time - think of spans and the q-Pascal triangle.

Recall from last time:



Here, X & Y are spans of finite sets.

$$Y = \left[\begin{array}{ccc} & \binom{n}{k}_F & \\ \binom{n}{k}_F \xleftarrow{id} & & \xrightarrow{\quad} \binom{n+1}{k}_F \\ & \searrow & \\ & V \subseteq F^n \cong F^n \oplus F & \end{array} \right] \quad (\text{NOT Pascal's } \Delta \text{ but a genuine function, hence a span})$$

It's awkward to treat it as a span of G -sets, since $GL(n, F)$ acts on F^n but $GL(n+1, F)$ acts on F^{n+1}

(though this is just as awkward as $F^n \hookrightarrow F^n \oplus F$)

We have a way / sort of around this, via the embedding

$$\varphi: GL(n, F) \hookrightarrow GL(n+1, F)$$

But a better way is to work with groupoids.

② ② A groupoid is a category ~~with~~ where all morphisms are invertible. (Any category \mathcal{C} has an underlying groupoid \mathcal{C}_0 , with the same objects, but only isomorphisms as morphisms.)

Thus, the skeleton of any groupoid is a disjoint union, & coproduct, of groupoids with one element/object.

Defn A group is a 1-object groupoid.

E.g. $(\text{Fin Set})_0 \cong \coprod_{n \geq 0} n!$, $(\text{Fin Vect}_F)_0 \cong \coprod_{n \geq 0} GL(n, F)$.

(b) We can thus switch from thinking about spans of G -sets to spans of groupoids, as follows:

Roughly speaking, a G -set S can be turned into a groupoid via the weak quotient $S//G$,

and such that given $\begin{cases} G\text{-set } S \\ G'\text{-set } S' \end{cases}$, $\varphi = \text{gp. hom. } G \rightarrow G'$,

and $f: S \rightarrow S'$ s.t.

$$\begin{array}{ccc} G \times S & \xrightarrow{\varphi \times \varphi} & G' \times S' \\ \text{act} \downarrow & \Downarrow & \downarrow \text{act}' \\ S & \xrightarrow{f} & S' \end{array}$$



all this should yield a functor $\mathbb{F}: S//G \rightarrow S'//G'$

(c) Let us first describe how to carry this out. Given a G -set S , how does one build a groupoid?

eg. $\mathbb{Z}/2\mathbb{Z} = G$ acts on $S = \begin{matrix} \circ & \circ \\ \longleftarrow & \longrightarrow \end{matrix} \begin{matrix} \circ \\ \circ \end{matrix}$
 and one now has $G = \{ \uparrow, \circ \circ \circ \}$

So, this is the idea in general:

- $S//G$ has
- objects = elements of S
 - morphisms = $[s \xrightarrow{f} g(s)] = G \times S$
 - composition = $(h, g(s)) \circ (g, s) = (hg, s)$
 - inverse = $(g, s)^{-1} = (g^{-1}, g(s))$

d) Let us work out ~~a~~ a specific example - for the case that we are interested in!

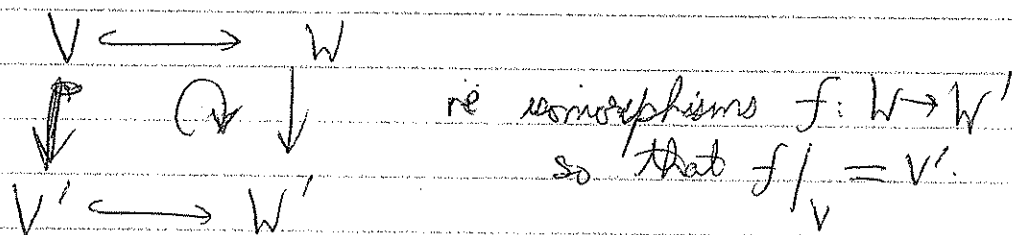
$GL(n, F)$ act on $\binom{n}{k}_F$. So, we get a groupoid

$\binom{n}{k}_F // GL(n, F) \rightarrow$ Objects are $V = k\text{-dim} \subseteq F^n$
 Morphisms $g: V \rightarrow V'$ are $g \in GL_n \cdot g|_V \rightarrow V'$.

In fact, this groupoid is equivalent to Flag $_{n,k}$

= ~~groupoid~~ groupoid whose objects are n -dim vector spaces (over F) equipped with a k -dim subsp.

and morphisms $f: (V \subseteq W) \rightarrow (V' \subseteq W')$ is



This is the groupoid of " n -dim vector spaces equipped with a k -dim subspaces"

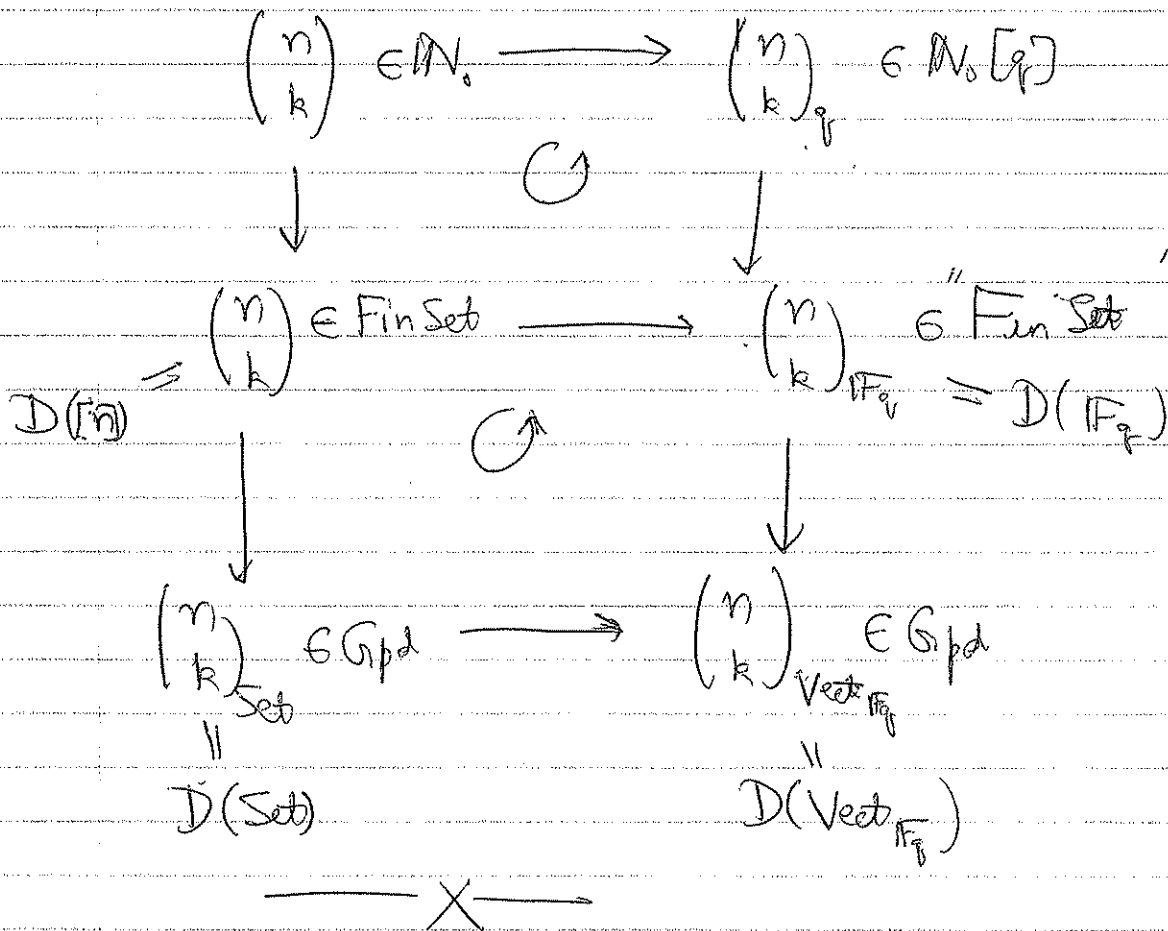
e) More generally, given $D = [n_1 + n_2 + \dots + n_k = n] = \text{flag}$,

$D(F^n) = \binom{n}{n_1, \dots, n_k}_F = D\text{-flags on } F^n$

Now, $D(F^n) // GL(n, F)$ is \cong the groupoid of vec spc over F equipped w/ D -flag

Let us give this the name $D(\text{Vect } F)$

(f) Now, we can apply D to various things:



(3) Next time, how $S \xrightarrow{\phi} S'$, $G \xrightarrow{\psi} G'$ st $\xrightarrow{\quad}$
 gives a functor $\mathbb{F} : S // G \rightarrow S' // G'$

and thus a span of G -sets gives a span of groupoids

The ultimate aim is to replace math / lit algebrae with
 spans and groupoids — the program called

groupoidification.