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1 (a) **Theorem** (k any field) Let G be a finite group, and X, Y be finite G-sets. Let $\hom(X,Y)$ be the set of isoclasses of spans of G-sets

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

(So, F_{ij} and "G is compatible")

$$\begin{tikzcd} & F_{ij} \arrow[dl] \arrow[dr] & \\ i & & j \end{tikzcd}$$

This is a finitely generated $\mathbb{N}$-module, and we have a map

$$\hom(X,Y) \longrightarrow \mathrm{Hom}_G(k^X, k^Y) = \{\text{intertwining operators } f: k^X \to k^Y$$

for any field k. The resulting map:

$$\hom(X,Y) \otimes_{\mathbb{N}} k \longrightarrow \mathrm{Hom}_G(k^X, k^Y)$$

is onto, but not necessarily 1-1.

(b) To state a better theorem, the "correct" decategorification needs to be used — not to come to the 1-category of isoclasses of G-spans, but to use spans of groupoids — which we'll see later in this course.

We'll ~~do~~ continue from where we left off last time — think of spans and the q-Pascal triangle.

Recall from last time:

$$\begin{tikzcd} & \binom{n}{k}_F \arrow[dl, "Y"'] \arrow[dr, "X"] & \\ \binom{n+1}{k}_F & & \binom{n+1}{k+1}_F \end{tikzcd}$$

Here, X & Y are spans of finite sets.

$$Y = \left[\begin{tikzcd} & \binom{n}{k}_F \arrow[dl, "id"'] \arrow[dr] \arrow[d] & \\ \binom{n}{k}_F & V \subseteq F^n \cong F^n \oplus F & \binom{n+1}{k}_F \end{tikzcd}\right]$$

(NOT Pascal's Δ but a genuine function, hence a span.)

It's awkward to treat it as a span of G-sets, since $GL(n, F)$ acts on F^n but $GL(n+1, F)$ acts on F^{n+1}

(though this is just as awkward as $F^n \hookrightarrow F^n \oplus F$!),

We have a way, sort of around this, via the embedding

$$\varphi: GL(n, F) \hookrightarrow GL(n+1, F).$$

But a better way is to work with groupoids.

② ⓐ A <u>groupoid</u> is a category ~~with~~ where all morphisms are invertible. (Any category $\mathcal{C}$ has an underlying groupoid $\mathcal{C}_0$, with the same objects, but only isomorphisms as morphisms.)

Thus, the skeleton of any groupoid is a disjoint union, or coproduct, of groupoids with one element/object.

Defn: A <u>group</u> is a 1-object groupoid.

E.g. $(\text{FinSet})_0 \simeq \coprod_{n \geq 0} n!$, $(\text{FinVect}_F)_0 \simeq \coprod_{n \geq 0} GL(n, F)$.

(b) We can thus switch from thinking about spans of G-sets to spans of groupoids, as follows:

Roughly speaking, a G-set S can be turned into a groupoid via the <u>weak quotient</u> $S//G$,

and such that given G-set S, G'-set S', $\varphi = $ gp hom : $G \to G'$,

and $\phi : S \to S'$ s.t.

$$\begin{tikzcd} G \times S \arrow[r, "\varphi \times \phi"] \arrow[d, "act"'] \arrow[dr, phantom, "\circlearrowleft"] & G' \times S' \arrow[d, "act'"] \\ S \arrow[r, "\phi"'] & S' \end{tikzcd}$$

$\downarrow$

all this should yield a <u>functor</u> $\Phi : S//G \longrightarrow S'//G'$

(c) Let us first describe how to carry this out. Given a G-set S, how does one build a groupoid?

e.g. $\mathbb{Z}/2\mathbb{Z} = G$ acts on $S =$

and one now has $G = \{$ $\}$

So, this is the idea in general:

$S//G$ has
- objects = elements of S
- morphisms $= [\, s \xrightarrow{g} g(s) \,] = G \times S$
- composition $= (h, g(s)) \cdot (g, s) = (hg, s)$
- inverse $= (g, s)^{-1} = (g^{-1}, gs)$

(d) Let us work out ~~th~~ a specific example - for the case that we are interested in!

$GL(n, F)$ act on $\binom{n}{k}_F$. So, we get a groupoid

$\binom{n}{k}_F // GL(n,F) \longrightarrow$ Objects are $V = k$-dim $\subseteq F^n$
Morphisms ~~are~~ : $V \to V'$ are
$g \in GL_n$: $g : V \to V'$.

In fact, this groupoid is equivalent to $\underline{Flag_{n,k}}$

= groupoid whose objects are n-dim vector spaces (over F) ~~equip~~ equipped with a k-dim subsp.

and morphisms $f : (V \subseteq W) \longrightarrow (V' \subseteq W')$ is

$$\begin{tikzcd} V \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & W \arrow[d] \\ V' \arrow[r, hook] & W' \end{tikzcd}$$

ie isomorphisms $f : W \to W'$ so that $f|_V = V'$.

This is the groupoid of "n-dim vector spaces equipped with a k-dim subspaces"!

(e) More generally, given $D = [n_1 + n_2 + \cdots n_k = n] =$ flag,

$D(F^n) = \binom{n}{n_1, \cdots n_k}_F$ = D-flags on F^n

Now, $D(F^n) // GL(n, F)$ is $\cong$ the groupoid of vec spc over F, equipped w/ D-flag

Let us give this the name $\underline{D(\mathrm{Vect}_F)}$

(f) Now, we can apply D to various things:

$$
\begin{tikzcd}
\binom{n}{k} \in \mathbb{N}_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \binom{n}{k}_q \in \mathbb{N}_0[q] \arrow[d] \\
D([n]) = \binom{n}{k} \in \mathrm{FinSet} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \binom{n}{k}_{\mathbb{F}_q} \in \text{"FinSet"} = D(\mathbb{F}_q) \arrow[d] \\
\binom{n}{k}_{\mathrm{Set}} \in \mathrm{Gpd} \arrow[r] \arrow[d, equal] & \binom{n}{k}_{\mathrm{Vect}_{\mathbb{F}_q}} \in \mathrm{Gpd} \arrow[d, equal] \\
D(\mathrm{Set}) & D(\mathrm{Vect}_{\mathbb{F}_q})
\end{tikzcd}
$$

—— X ——

(3) $\underline{\text{Next time}}$: how $S \xrightarrow{\phi} S'$, $G \xrightarrow{\varphi} G'$ s.t.
gives a functor $\Phi : S/\!/G \to S'/\!/G'$

and thus a span of G-sets gives a span of groupoids

The ultimate aim is to replace math / lin algebra with spans and groupoids — the program called

$\boxed{\text{groupoidification}}$