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(1) $\boxed{\text{Theorem}}$ The ~~Vect~~ [VECT]-enriched category

(a) (Finite G-sets, Hecke operators between their Free$_{\mathbb{C}}$'s)

is the decategorification of the [CAT]-enriched category

(Finite G-sets, Category of spans)

(b) John B- ~~stated the~~ theorem using the wrong kind of decategori
Also, he worked over $\mathbb{N}_0$; we'll work over $\mathbb{C}$.

(c) We first explain what enrichment is, and how our version of "decategorification" relates to it.

If $\mathcal{C}$ is a category and $\mathcal{D}$ is a category with some notion $\otimes$ in it, ~~then~~ a ($\mathcal{D}$-enriched category $\mathcal{C}_{\mathcal{D}}$) is

a category whose objects are ~~all~~ objects of $\mathcal{C}$ (all of them?)

and whose Hom-sets are now objects in $\mathcal{D}$.

So $\mathcal{D}$ needs to be a monoidal category

The reason $\mathcal{D}$ should have some $\otimes$ is so ~~that~~ we c
compose morphisms in $\mathcal{C}_{\mathcal{D}}$. Moreover, composition in $\mathcal{C}$ ~~~~ is a morphism in $\mathcal{D}$!

e.g. $\mathcal{D} = \text{Vect} \Rightarrow$ usual $\otimes = \otimes_{\mathbb{C}}$
$\mathcal{D} = \text{Cat} \Rightarrow$ direct product of categories.

(d) How is decategorification related to enrichment?

~~Pf~~ Given a $\otimes$-functor between $\otimes$ categories $\mathcal{A}, \mathcal{B}$,

we can define $P(\underline{DECAT}) : \mathcal{A}$-enriched CAT $\longrightarrow \mathcal{B}$-enriched CAT

& given any $\mathcal{C}$, $\mathcal{C}_{\mathcal{A}} \to \mathcal{C}_{\mathcal{B}}$

($\mathcal{C}_{\mathcal{B}}$ = Same objects as in $\mathcal{C}_{\mathcal{A}}$, but $Hom_{\mathcal{B}} = DECAT(Hom_{\mathcal{A}})$)

(Of course, we also only work with "more complex" $\longrightarrow$ "less complex",
but morally, this construction can be done more generally :)

e) $\boxed{\text{Example}}$: Say $\underline{1}$ = Category with 1 object & 1 morphism.

Then one has $SET \xrightarrow{D} \underline{1}$

all sets $\longrightarrow (*)$

all maps $\longrightarrow (id_*)$.

What is $\mathcal{C}_{SET} \xrightarrow{P(D)} \mathcal{C}_{\underline{1}}$?

<u>ANS</u>: $\mathcal{C}_{\underline{1}}$ has same # of objects as $\mathcal{C}_{SET} = \mathcal{C}$.

Set-enriched Cat's are just usual categories!

But all objects have only 1 map between them,

$\boxed{\text{Lemma}}$ All objects are isomorphic.

$\boxed{\text{Pf}}$ $\mu_{XY} : X \longrightarrow Y \xrightarrow{\mu_{YX}} X$ must be id_X $\forall X$. ▨

(f) Conclusion : (Isoclasses of small $\mathbb{1}$-enriched categories)

$\Updownarrow$

(Set).

(Equivalence classes of (small) $\mathbb{1}$-enriched categories) $\Leftrightarrow \{0, 1\}$

The empty one!

$\{0,1\} = TV =$ Truth Values.

(g) A TV-enriched category means every morphism set has the choice to be empty or nonempty, and every nonempty set just has one morphism in it.

Moreover, the category $\mathcal{C}_{TV}$ is itself a category! So $|\mathrm{Hom}(X,X)| = 1 \; \forall X$, etc.

Thus, $\mathcal{C}_{TV}$ is just a $\boxed{\text{partial order on } Ob(\mathcal{C})}$!!

—— X ——

(2) We want $\mathcal{C}_{CAT} \rightarrow \mathcal{C}_{\mathbb{C}\text{-Vect}}$.

(a) Actually, we ~~do~~ use D: "nice ~~CAT~~" $\longrightarrow$ $\mathbb{C}$-VECT.

to get $P(D)$. ~~...~~ "Nice" means so nice, that it's actually a groupoid in "disguise".

That is it's the ~~categorification~~ category of G-sets for some groupoid G.

(b) Groupoid theory is not very popular because people ~~get~~ break up the groupoid into its isoclasses and then just ~~study~~ its skeleton, so this comes back to group theory!

Also unpopular is the zeroth homology of groups: as top. spaces, gps. have ~~...~~ just 1-component, and

$$H_0(\text{top space } X; \mathbb{R}) = \mathbb{R}^{\#\text{ components}}$$

But now, we can do $H_0(\text{groupoid}) = \mathbb{R}^{\#\text{ isoclasses in } G}$, for G a groupoid!

What makes this process even more interesting is the notion of "transfers", for (co)homology theories.

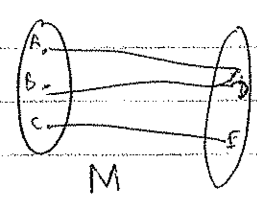
(c) Generally / philosophically, homology is a covariant functor, and cohomology is a contravariant functor.

But sometimes, we get ~~that~~ the other sort of variance too!

e.g. $X \xleftarrow{M} Y \xrightarrow{N} Z$ gives ~~the~~ usual $N_*: H_0(Y) \to H_0(Z)$

but also the more interesting $M^!: H_0(X) \to H_0(Y)$!

(d) Simple examples involving transfers:



$S \xrightarrow{M} T$

	A	B	C
E	1	1	
F			1

~~$H_0(S) \leftarrow H_0(T)$~~

~~The~~

Then

$$H_0(S) \xrightarrow{M_*} H_0(T)$$

$$\alpha A + \beta B + \gamma C \mapsto (\alpha + \beta) D + \gamma E$$

i.e.

$$\begin{array}{c|ccc} & A & B & C \\ \hline D & 1 & 1 & 0 \\ E & 0 & 0 & 1 \end{array}$$

has the rather interesting "transfer" map $M^! = (M_*)^T$!

$$\alpha D + \beta E \mapsto \alpha(A+B) + \beta C$$

("summing over the fiber")

(e) In fact, ~~this~~ is exactly how spans give rise to matrices! Both sides give us numbers....

(f) But the really interesting thing comes when we start with groupoids! We then turn them into topological spaces and thence, on into vector spaces.

Spans ~~but~~ turn into ~~a~~ linear operators, and this happens by means of the "transfer trick".

This is what we ~~had~~ had called de-groupoid-ification although one can also call it the zeroth homology for groupoids.

This is the DECAT we use, as said above!